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10 points 1. Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case).

(1) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be otained by performing an elementary row operation on B.

(2) The rank of a matrix is equal to the number of its nonzero columns.

(3) The product of two matrices always has rank equal to the lesser of the ranks of the two matrices.

(4) Elementary row operations perserve rank.

(5) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.

(6) An $n \times n$ matrix having rank n is invertible.

(7) Any homogeneous system of linear equations has at least one solution.

(8) If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.

(9) The solution set of any system of m linear equations in n unknowns is a subspace of F^n.

(10) If A is an $n \times n$ matrix with rank n, then the reduced row echelon form of A is I_n.

5 points 2. Find the inverse of each of the following elementary matrices

$$\text{(a)} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(b)} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{(c)} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

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7.5 points 3. Let

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{pmatrix}.$$

Express A^{-1} as a product of elementary matrices.

10 points 4. Let A be $m \times n$ with $m < n$. Prove that the system $Ax = 0$ has a nonzero solution.

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7.5 points 5. Describe the solution set of $2x_1 - 4x_2 - 4x_3 = 0$; compare it to the solution set $2x_1 - 4x_2 - 4x_3 = 6$.

10 points 6. (**BONUS PROBLEM**) Let A be $m \times n$, and P, Q invertible of sizes $m \times m$, $n \times n$. Prove that

(a) $\text{rank}(AQ) = \text{rank}(A)$

(b) $\text{rank}(PA) = \text{rank}(A)$

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(1) True. From $B = EA$ with E an elementary matrix, it follows that $A = E^{-1}B$ where the inverse E^{-1} is also an elementary matrix.

(2) False. For example, the rank of $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ is 1, not equal to the number of its nonzero columns.

(3) False. For example, for $A = (1, 0)$ and $B = (0, 1)^t$, both having rank 1, the product $AB = (0)$ has rank 0. Theorem 3.7 implies that $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.

(4) True. From Corollary in Page 153, elementary row operations perserve rank.

(5) True. From Corollary 2 in Page 158, $\text{rank}(A) = \dim(\text{row space of } A) = \dim(\text{column space of } A)$.

(6) True. From Theorem 2.5, $L_A : F^n \to F^n$ is invertible if and only if $\text{rank}(L_A) = \dim(F^n)$, i.e., $\text{rank}(A) = n$. From Corollary 2 in Page 102, A is invertible if and only if L_A is invertible. Then A is invertible if and

(7) True. Any homogeneous system of linear equations has zero as a solution.

(8) False. For example, the system that $0x = 1$ has no solution while the corresponding homogeneous system $0x = 0$ has a solution.

(9) False. For example, the solution set of the system $x = 1$ is not a subspace of F.

(10) True.

Problem 2.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$.

Problem 3. Perform the row operations to reduce the matrix A to the identity matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 0 & -3 & 13 \end{bmatrix} \quad \text{with} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -4 \\ 0 & -3 & 13 \end{bmatrix} \quad \text{with} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with} \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with} \quad E_5 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad \text{with} \quad E_6 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In matrix form, we have

$$E_6(E_5(E_4(E_3(E_2(E_1A))))) = I_3.$$

Therefore, by the uniqueness of the inverse matrix of A, we have

$$A^{-1} = E_6E_5E_4E_3E_2E_1.$$

Problem 4.
Suppose that $m < n$. Then $\operatorname{rank}(A) = \operatorname{rank}(L_A) \leq m$. Hence

$$\dim(N(L_A)) = n - \operatorname{rank}(L_A) \geq n - m > 0,$$

Since $\dim(N(L_A)) > 0$, $N(L_A) \neq \{0\}$. Then there exists a nonzero vector $s \in N(L_A)$; so s is a nonzero solution to $Ax = 0$.

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Problem 5. Note that the corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 0$ is

$$\begin{pmatrix} 2 & -4 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -2 & 0 \end{pmatrix}$$

The vector form of the solution is

$$v = \begin{pmatrix} 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

The corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 6$ is

$$\begin{pmatrix} 2 & -4 & -4 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -2 & 3 \end{pmatrix}$$

The vector form of the solution is

$$v = \begin{pmatrix} 2x_2 + 2x_3 + 3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 6. (BONUS PROBLEM)

(a) First observe that

$$\mathrm{R}(L_{AQ}) = \mathrm{R}(L_A L_Q) = L_A L_Q(F^n) = L_A(L_Q(F^n)) = L_A(F^n) = \mathrm{R}(L_A)$$

since L_Q is onto. Therefore,

$$\mathrm{rank}(AQ) = \dim(\mathrm{R}(L_{AQ})) = \dim(\mathrm{R}(L_A)) = \mathrm{rank}(A).$$

(b) Observe that

$$\mathrm{R}(L_{PA}) = \mathrm{R}(L_P L_A) = L_P L_A(F^n) = L_P(L_A(F^n)), \quad \mathrm{R}(L_A) = L_A(F^n).$$

Note that, since P is invertible, L_P is an isomorphism from F^n to F^n. From problem 2.4.17,

$$\dim(L_P(L_A(F^n))) = \dim(L_A(F^n)).$$

Therefore, we have

$$\mathrm{rank}(PA) = \dim(\mathrm{R}(L_{PA})) = \dim(L_P(L_A(F^n))) = \dim(L_A(F^n)) = \dim(\mathrm{R}(L_A)) = \mathrm{rank}(A).$$

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.