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10 points 1. Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case).

(1) If u and v are vectors in $\mathbb{R}^2$ emanating from the origin, then the area of the parallelogram having u and v as adjacent sides is $\det \begin{pmatrix} u \\ v \end{pmatrix}$.

(2) The function $\det : M_{n\times n}(F) \to F$ is a linear transformation.

(3) If B is a matrix obtained from a square matrix A by interchanging any two rows, then $\det(B) = -\det(A)$.

(4) If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $\det(B) = \det(A)$.

(5) If B is a matrix obtained from a square matrix A by adding k times row i to row j, then $\det(B) = k\det(A)$.

(6) If $A \in M_{n\times n}(F)$ has rank n, then $\det(A) = 0$.

(7) The determinant of a square matrix may be computed by expanding the matrix along any row or column.

(8) If two rows or columns of A are identical, then $\det(A) = 0$.

(9) The determinant of a lower triangular $n \times n$ matrix is the product of its diagonal entries.

(10) A matrix A is invertible if and only if $\det(A) \neq 0$.

10 points 2. Compute the determinant of each of the following matrices

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 3 & 4 \\ 2 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

10 points 3. Prove that if $A, B \in M_n(F)$ are similar, then $\det(A) = \det(B)$.

10 points 4. Suppose $M \in M_n(F)$ is skew-symmetric, i.e., $M^t = -M$. Show that if n is odd, then M is not invertible. What if n is even?

10 points 5. (**BONUS PROBLEM**) Suppose that $M \in M_n(F)$ can be written in the block upper triangular form

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A \in M_k(F)$ and $C \in M_{n-k}(F)$. Prove that

$$\det(M) = \det(A)\det(C).$$

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Problem 1.

(1) False. The area of the parallelogram having u and v as adjacent sides is $\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|$ (Page 204).

(2) Flase. The function $\det : M_{n\times n}(F) \to F$ is a linear transformation of each row when the remaining rows are held fixed (Theorem 4.3 in Page 212).

(3) True. If B is a matrix obtained from a square matrix A by interchanging any two rows, then $\det(B) = -\det(A)$ (Theorem 4.5 in Page 216).

(4) False. If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar k, then $\det(B) = k\det(A)$ (Page 217).

(5) False. If B is a matrix obtained from a square matrix A by adding k times row i to row j, then $\det(B) = \det(A)$ (Theorem 4.6 in Page 216).

(6) False. If $A \in M_{n\times n}(F)$ has rank n, then $\det(A) \neq 0$ (Corollary in Page 223).

(7) True. The determinant of a square matrix may be computed by expanding the matrix along any row or column (Theorems 4.4 and 4.8 in Page 224).

(8) True. If two rows or columns of A are identical, then $\det(A) = 0$ (Corollary in Page 215 and Theorem 4.8 in Page 224).

(9) True. The determinant of a lower triangular $n \times n$ matrix is the product of its diagonal entries (Exercise 23 in Page 222 and Theorem 4.8 in Page 224).

(10) True. A matrix A is invertible if and only if $\det(A) \neq 0$ (Corollary in Page 223).

Problem 2.

(a)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 2 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = -5(1)(1)(2) = -10$$

(b)

$$\det\begin{pmatrix} 1 & 0 & 3 & 4 \\ 2 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}\det\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} = (1)(2)(2\cdot 5 - 3\cdot 1) = 14.$$

(c)

$$\det\begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{pmatrix} = (2)(1)(-3)(4) = -24.$$

(d)

$$\det\begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = -\det\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = -(2)(1)(3)(4) = -24.$$

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Problem 3.

If A and B are similar, then

$$A = Q^{-1}BQ$$

for some invertible Q. Taking determinants gives

$$\begin{aligned}\det(A) &= \det(Q^{-1}BQ) = \det(Q^{-1})\det(B)\det(Q)\\ &= \frac{1}{\det(Q)}\det(B)\det(Q) = \det(B).\end{aligned}$$

Problem 4.

We know that

$$\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M).$$

If n is odd, then $\det(M) = -det(M)$, so $\det(M) = 0$, and M is not invertible. However, this equation tells us nothing if n is even. To fully answer this question, we must include examples of skew symmetric matrices which are invertible and non-invertible for n even. For $n = 2$, both

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are skew symmetric; the first is not invertible, and the second is invertible. Now for $n > 2$, we get examples by taking block diagonal matrices using the above examples.

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Problem 5. (BONUS PROBLEM)

We proceed by induction on n.

- $n = 2$: Obvious as
$$\det(M) = \det \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = ac.$$

- $n - 1 \Rightarrow n$: We take the determinant by expanding along the first column of M. Let $\tilde{M}_{ij}$ be the matrix obtained from M by deleting the ith row and jth column. First, note that $M_{i1} = 0$ for all $i > k$. For $i \le k$, $M_{i1} = A_{i1}$ and
$$\det \tilde{M}_{i1} = \det \begin{pmatrix} \tilde{A}_{i1} & \tilde{B} \\ 0 & C \end{pmatrix} = \det(\tilde{A}_{i1}) \det(C)$$
by the induction hypothesis as $\tilde{M}_{i1}$ is block upper triangular. Then
$$\begin{aligned} \det(M) &= \sum_{i=1}^{n} (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1}) = \sum_{i=1}^{k} (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1}) \\ &= \left(\sum_{i=1}^{k} (-1)^{i+1} A_{i1} \det(\tilde{A}_{i1}) \right) \det(C) = \det(A) \det(C). \end{aligned}$$

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.