1. Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case).

(1) If \( u \) and \( v \) are vectors in \( \mathbb{R}^2 \) emanating from the origin, then the area of the parallelogram having \( u \) and \( v \) as adjacent sides is \( \det \begin{pmatrix} u \\ v \end{pmatrix} \).

(2) The function \( \det : M_{n \times n}(F) \to F \) is a linear transformation.

(3) If \( B \) is a matrix obtained from a square matrix \( A \) by interchanging any two rows, then \( \det(B) = -\det(A) \).

(4) If \( B \) is a matrix obtained from a square matrix \( A \) by multiplying a row of \( A \) by a scalar, then \( \det(B) = \det(A) \).

(5) If \( B \) is a matrix obtained from a square matrix \( A \) by adding \( k \) times row \( i \) to row \( j \), then \( \det(B) = k \det(A) \).

(6) If \( A \in M_{n \times n}(F) \) has rank \( n \), then \( \det(A) = 0 \).

(7) The determinant of a square matrix may be computed by expanding the matrix along any row or column.

(8) If two rows or columns of \( A \) are identical, then \( \det(A) = 0 \).

(9) The determinant of a lower triangular \( n \times n \) matrix is the product of its diagonal entries.

(10) A matrix \( A \) is invertible if and only if \( \det(A) \neq 0 \).

2. Compute the determinant of each of the following matrices

(a) \[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & 5 & 0 & 0 \\
2 & 7 & 6 & 10 \\
2 & 9 & 7 & 11
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 & 0 & 3 & 4 \\
2 & 2 & 1 & 5 \\
0 & 0 & 2 & 1 \\
0 & 0 & 3 & 5
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
2 & 3 & 4 & 5 \\
0 & 1 & 2 & 3 \\
0 & 0 & -3 & 5 \\
0 & 0 & 0 & 4
\end{pmatrix}
\]

(d) \[
\begin{pmatrix}
0 & 0 & 3 & 0 \\
0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 4
\end{pmatrix}
\]

3. Prove that if \( A, B \in M_n(F) \) are similar, then \( \det(A) = \det(B) \).

4. Suppose \( M \in M_n(F) \) is skew-symmetric, i.e., \( M^t = -M \). Show that if \( n \) is odd, then \( M \) is not invertible. What if \( n \) is even?

5. (BONUS PROBLEM) Suppose that \( M \in M_n(F) \) can be written in the block upper triangular form

\[
M = \begin{pmatrix}
A & B \\
0 & C
\end{pmatrix}
\]

where \( A \in M_k(F) \) and \( C \in M_{n-k}(F) \). Prove that

\[
\det(M) = \det(A) \det(C).
\]
Problem 1.

(1) False. The area of the parallelogram having $u$ and $v$ as adjacent sides is $|\det(u, v)|$ (Page 204).

(2) False. The function $\det : M_{n \times n}(F) \to F$ is a linear transformation of each row when the remaining rows are held fixed (Theorem 4.3 in Page 212).

(3) True. If $B$ is a matrix obtained from a square matrix $A$ by interchanging any two rows, then $\det(B) = -\det(A)$ (Theorem 4.5 in Page 216).

(4) False. If $B$ is a matrix obtained from a square matrix $A$ by multiplying a row of $A$ by a scalar $k$, then $\det(B) = k \det(A)$ (Page 217).

(5) False. If $B$ is a matrix obtained from a square matrix $A$ by adding $k$ times row $i$ to row $j$, then $\det(B) = \det(A)$ (Theorem 4.6 in Page 216).

(6) False. If $A \in M_{n \times n}(F)$ has rank $n$, then $\det(A) \neq 0$ (Corollary in Page 223).

(7) True. The determinant of a square matrix may be computed by expanding the matrix along any row or column (Theorems 4.4 and 4.8 in Page 224).

(8) True. If two rows or columns of $A$ are identical, then $\det(A) = 0$ (Corollary in Page 215 and Theorem 4.8 in Page 224).

(9) True. The determinant of a lower triangular $n \times n$ matrix is the product of its diagonal entries (Exercise 23 in Page 222 and Theorem 4.8 in Page 224).

(10) True. A matrix $A$ is invertible if and only if $\det(A) \neq 0$ (Corollary in Page 223).

Problem 2.

(a) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{vmatrix} = 5(1)(1)(2) = -10.$

(b) $\begin{vmatrix} 1 & 0 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} = \det \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \det \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} = (1)(2)(2 \cdot 5 - 3 \cdot 1) = 14.$

(c) $\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} = (2)(1)(-3)(4) = -24.$

(d) $\begin{vmatrix} 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = -\det \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = -(2)(1)(3)(4) = -24.$
Problem 3.
If $A$ and $B$ are similar, then

$$A = Q^{-1}BQ$$

for some invertible $Q$. Taking determinants gives

$$\det(A) = \det(Q^{-1}BQ) = \det(Q^{-1}) \det(B) \det(Q)$$

$$= \frac{1}{\det(Q)} \det(B) \det(Q) = \det(B).$$

Problem 4.
We know that

$$\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M).$$

If $n$ is odd, then $\det(M) = -\det(M)$, so $\det(M) = 0$, and $M$ is not invertible. However, this equation tells us nothing if $n$ is even. To fully answer this question, we must include examples of skew symmetric matrices which are invertible and non-invertible for $n$ even. For $n = 2$, both

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are skew symmetric; the first is not invertible, and the second is invertible. Now for $n > 2$, we get examples by taking block diagonal matrices using the above examples.
Problem 5. (BONUS PROBLEM)

We proceed by induction on \(n\).

- \(n = 2\): Obvious as
  \[
  \det(M) = \det\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = ac.
  \]

- \(n - 1 \Rightarrow n\): We take the determinant by expanding along the first column of \(M\). Let \(\tilde{M}_{ij}\) be the matrix obtained from \(M\) by deleting the \(i\)th row and \(j\)th column. First, note that \(M_{i1} = 0\) for all \(i > k\). For \(i \leq k\), \(M_{i1} = A_{i1}\) and

  \[
  \det \tilde{M}_{i1} = \det\begin{pmatrix} \tilde{A}_{i1} & \tilde{B} \\ 0 & C \end{pmatrix} = \det(\tilde{A}_{i1}) \det(C)
  \]

  by the induction hypothesis as \(\tilde{M}_{i1}\) is block upper triangular. Then

  \[
  \det(M) = \sum_{i=1}^{n} (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1}) = \sum_{i=1}^{k} (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1})
  \]

  \[
  = \left( \sum_{i=1}^{k} (-1)^{i+1} A_{i1} \det(\tilde{A}_{i1}) \right) \det(C) = \det(A) \det(C).
  \]