Math 4377/6308 Advanced Linear Algebra
1.4 Linear Combinations & Systems of Linear Equations

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/~jiwenhe/math4377
1.4 Linear Combinations & Systems of Linear Equations

- Linear Combinations: Definition
- Linear Combinations of Vectors in $\mathbb{R}^2$
- Linear Combinations and Vector Equation
- Solving a System of Linear Equations by Row Eliminations
- Span of a Set of Vectors: Definition
- Span of a Set of Vectors in $\mathbb{R}^2$ and in $\mathbb{R}^3$
- A Shortcut for Determining Subspaces
- Spanning Sets
Definition

Let $V$ be a vector space and $S$ a nonempty subset of $V$. A vector $v \in V$ is called a **linear combination** of vectors of $S$ if there exist a finite number of vectors $u_1, u_2, \ldots, u_n$ in $S$ and scalars $a_1, a_2, \ldots, a_n$ in $F$ such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$ 

In this case we also say that $v$ is a **linear combination** of $u_1, u_2, \ldots, u_n$ and call $a_1, a_2, \ldots, a_n$ the **coefficients** of the linear combination.

Note that $0v = 0$ for each $v \in V$, so the zero vector is a linear combination of any nonempty subset of $V$. 
Parallelogram Rule for Addition of Two Vectors

If \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^2 \) are represented as points in the plane, then \( \mathbf{u} + \mathbf{v} \) corresponds to the fourth vertex of the parallelogram whose other vertices are \( \mathbf{0}, \mathbf{u} \) and \( \mathbf{v} \).

Geometric Description of \( \mathbb{R}^2 \)

Vector \( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) is the point \( (x_1, x_2) \) in the plane. \( \mathbb{R}^2 \) is the set of all points in the plane.

Example

Let \( \mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) and \( \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).

Graphs of \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{u} + \mathbf{v} \) are:
Example

Let \( \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

Express \( \mathbf{u}, 2\mathbf{u}, \) and \( \frac{-3}{2}\mathbf{u} \) on a graph.
Example

Let \( \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) and \( \mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \). Express each of the following as a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \):

\[
\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}
\]
Linear Combinations: Example

Example

Let \( \mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \), \( \mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} \), \( \mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \), and \( \mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix} \).

Determine if \( \mathbf{b} \) is a linear combination of \( \mathbf{a}_1 \), \( \mathbf{a}_2 \), and \( \mathbf{a}_3 \).

Solution: Vector \( \mathbf{b} \) is a linear combination of \( \mathbf{a}_1 \), \( \mathbf{a}_2 \), and \( \mathbf{a}_3 \) if we can find weights \( x_1 \), \( x_2 \), \( x_3 \) such that

\[
x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3 = \mathbf{b}.
\]

Vector Equation (fill-in):

\[
x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}.
\]
Corresponding System:

\[ x_1 + 4x_2 + 3x_3 = -1 \]
\[ 2x_2 + 6x_3 = 8 \]
\[ 3x_1 + 14x_2 + 10x_3 = -5 \]

Corresponding Augmented Matrix:

\[
\begin{bmatrix}
1 & 4 & 3 & -1 \\
0 & 2 & 6 & 8 \\
3 & 14 & 10 & -5
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

\[ x_1 = \ldots \]
\[ x_2 = \ldots \]
\[ x_3 = \ldots \]
Review of the last example: \( a_1, a_2, a_3 \) and \( b \) are columns of the augmented matrix

\[
\begin{bmatrix}
1 & 4 & 3 & -1 \\
0 & 2 & 6 & 8 \\
3 & 14 & 10 & -5 \\
\end{bmatrix}
\]

\[ \uparrow \uparrow \uparrow \uparrow \]

\( a_1 \ a_2 \ a_3 \ b \)

Solution to

\[ x_1a_1 + x_2a_2 + x_3a_3 = b \]

is found by solving the linear system whose augmented matrix is

\[
\begin{bmatrix}
a_1 & a_2 & a_3 & b \\
\end{bmatrix}
\]
Vector Equation

A vector equation

\[ x_1a_1 + x_2a_2 + \cdots + x_na_n = b \]

has the same solution set as the linear system whose augmented matrix is

\[
\begin{bmatrix}
a_1 & a_2 & \cdots & a_n & b \\
\end{bmatrix}
\]

In particular, \( b \) can be generated by a linear combination of \( a_1, a_2, \ldots, a_n \) if and only if there is a solution to the linear system corresponding to the augmented matrix.
Solving a System of Linear Equations

Example

Solving a System in Matrix Form

\[
\begin{align*}
    x_1 - 2x_2 &= -1 \\
    -x_1 + 3x_2 &= 3
\end{align*}
\]

\[
\begin{bmatrix}
    1 & -2 & -1 \\
    -1 & 3 & 3
\end{bmatrix}
\text{(augmented matrix)}
\]

\[
\begin{align*}
    x_1 - 2x_2 &= -1 \\
    x_2 &= 2
\end{align*}
\]

\[
\begin{bmatrix}
    1 & -2 & -1 \\
    0 & 1 & 2
\end{bmatrix}
\]

\[
\begin{align*}
    x_1 &= 3 \\
    x_2 &= 2
\end{align*}
\]

\[
\begin{bmatrix}
    1 & 0 & 3 \\
    0 & 1 & 2
\end{bmatrix}
\]
Row Operations

**Elementary Row Operations**

1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

**Row Equivalent Matrices**

Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

**Fact about Row Equivalence**

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
Solving a System by Row Eliminations: Example

Example (Row Eliminations to a Triangular Form)

\[
\begin{align*}
  x_1 &- 2x_2 + x_3 = 0 \\
  2x_2 &- 8x_3 = 8 \\
-4x_1 &+ 5x_2 + 9x_3 = -9
\end{align*}
\]

\[
\begin{pmatrix}
  1 & -2 & 1 & 0 \\
  0 & 2 & -8 & 8 \\
-4 & 5 & 9 & -9
\end{pmatrix}
\]

\[
\begin{align*}
  x_1 &- 2x_2 + x_3 = 0 \\
  2x_2 &- 8x_3 = 8 \\
-3x_2 &+ 13x_3 = -9
\end{align*}
\]

\[
\begin{pmatrix}
  1 & -2 & 1 & 0 \\
  0 & 2 & -8 & 8 \\
0 & -3 & 13 & -9
\end{pmatrix}
\]

\[
\begin{align*}
  x_1 &- 2x_2 + x_3 = 0 \\
  x_2 &- 4x_3 = 4 \\
-3x_2 &+ 13x_3 = -9
\end{align*}
\]

\[
\begin{pmatrix}
  1 & -2 & 1 & 0 \\
  0 & 1 & -4 & 4 \\
0 & -3 & 13 & -9
\end{pmatrix}
\]

\[
\begin{align*}
  x_1 &- 2x_2 + x_3 = 0 \\
  x_2 &- 4x_3 = 4 \\
x_3 & = 3
\end{align*}
\]

\[
\begin{pmatrix}
  1 & -2 & 1 & 0 \\
  0 & 1 & -4 & 4 \\
0 & 0 & 1 & 3
\end{pmatrix}
\]
Example (Row Eliminations to a Diagonal Form)

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
x_3 &= 3
\end{align*}
\]

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

\[
\begin{align*}
x_1 - 2x_2 &= -3 \\
x_2 &= 16 \\
x_3 &= 3
\end{align*}
\]

\[
\begin{bmatrix}
1 & -2 & 0 & -3 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

\[
\begin{align*}
x_1 &= 29 \\
x_2 &= 16 \\
x_3 &= 3
\end{align*}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 29 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

Solution: (29, 16, 3)
Example (Check the Answer)

Is \((29, 16, 3)\) a solution of the original system?

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9
\end{align*}
\]

\[
\begin{align*}
(29) - 2(16) + (3) &= 29 - 32 + 3 = 0 \\
2(16) - 8(3) &= 32 - 24 = 8 \\
-4(29) + 5(16) + 9(3) &= -116 + 80 + 27 = -9
\end{align*}
\]
Example

Let \( \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \).

Label the origin together with \( \mathbf{v}, 2\mathbf{v} \) and \( 1.5\mathbf{v} \) on the graph.

\( \mathbf{v}, 2\mathbf{v} \) and \( 1.5\mathbf{v} \) all lie on the same line.

\textbf{Span}\{\mathbf{v}\} is the set of all vectors of the form \( c\mathbf{v} \).

Here, \( \text{Span}\{\mathbf{v}\} = \) a line through the origin.
Example

Label $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$ and $3\mathbf{u} + 4\mathbf{v}$ on the graph.

$\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$ and $3\mathbf{u} + 4\mathbf{v}$ all lie in the same plane.

$\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is the set of all vectors of the form $x_1\mathbf{u} + x_2\mathbf{v}$. Here, $\text{Span}\{\mathbf{u}, \mathbf{v}\} = \text{a plane through the origin}$. 
Span of a Set of Vectors: Definition

**Span of a Set of Vectors**
Suppose \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \) are in \( \mathbb{R}^n \); then

\[
\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} = \text{set of all linear combinations of } \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p.
\]

**Span of a Set of Vectors (Stated another way)**

\[
\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} \text{ is the collection of all vectors that can be written as } \\
x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_p \mathbf{v}_p
\]

where \( x_1, x_2, \ldots, x_p \) are scalars.
Example

Let \( \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) and \( \mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \).

(a) Find a vector in \( \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \).

(b) Describe \( \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \) geometrically.
Spanning Sets in $\mathbb{R}^3$

Example

$\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$

$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{v}_1\}$

$= \text{Span}\{\mathbf{v}_2\}$

(line through the origin)
Example

Let \( \mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \) and \( \mathbf{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} \).

Is \( \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \) a line or a plane?

\( \mathbf{v}_2 \) is not a multiple of \( \mathbf{v}_1 \)

\( \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \) plane through the origin
Spanning Sets

Example

Let \( A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} \) and \( b = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix} \). Is \( b \) in the plane spanned by the columns of \( A \)?

Solution: Do \( x_1 \) and \( x_2 \) exist so that

\[
\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} x_2 = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}
\]

Corresponding augmented matrix:

\[
\begin{bmatrix} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{bmatrix}
\]

So \( b \) is not in the plane spanned by the columns of \( A \).
A Shortcut for Determining Subspaces

Theorem (1)

If \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) are in a vector space \( V \), then \( \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \) is a subspace of \( V \).

**Proof:** In order to verify this, check properties a, b and c of definition of a subspace.

a. \( \mathbf{0} \) is in \( \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \) since

\[
\mathbf{0} = \underbrace{0}_{\mathbf{0}} \mathbf{v}_1 + \underbrace{0}_{\mathbf{0}} \mathbf{v}_2 + \cdots + \underbrace{0}_{\mathbf{0}} \mathbf{v}_p
\]

b. To show that \( \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \) closed under vector addition, we choose two arbitrary vectors in \( \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \):

\[
\mathbf{u} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_p \mathbf{v}_p
\]

and

\[
\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p.
\]
Then
\[ u + v = (a_1v_1 + a_2v_2 + \cdots + a_pv_p) + (b_1v_1 + b_2v_2 + \cdots + b_pv_p) \]
\[ = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \cdots + (a_p + b_p)v_p. \]

So \( u + v \) is in Span\( \{v_1, \ldots, v_p\} \).

c. To show that Span\( \{v_1, \ldots, v_p\} \) closed under scalar multiplication, choose an arbitrary number \( c \) and an arbitrary vector in Span\( \{v_1, \ldots, v_p\} \) :
\[ v = b_1v_1 + b_2v_2 + \cdots + b_pv_p. \]
Then

\[ cv = c ( b_1 v_1 + b_2 v_2 + \cdots + b_p v_p ) \]

\[ = v_1 + v_2 + \cdots + v_p \]

So \( cv \) is in \( \text{Span}\{v_1, \ldots, v_p\} \).

Since properties a, b and c hold, \( \text{Span}\{v_1, \ldots, v_p\} \) is a subspace of \( V \).
1.4 Linear Combinations

Determining Subspaces: Recap

Recap

1. To show that $H$ is a subspace of a vector space, use Theorem 1.

2. To show that a set is not a subspace of a vector space, provide a specific example showing that at least one of the axioms a, b or c (from the definition of a subspace) is violated.
Example

Is \( V = \{(a + 2b, 2a - 3b) : a \text{ and } b \text{ are real}\} \) a subspace of \( \mathbb{R}^2 \)? Why or why not?

Solution: Write vectors in \( V \) in column form:

\[
\begin{bmatrix}
a + 2b \\
2a - 3b
\end{bmatrix}
= \begin{bmatrix}
a \\
2a
\end{bmatrix} + \begin{bmatrix}
2b \\
-3b
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 \\
2
\end{bmatrix} + \begin{bmatrix}
2 \\
-3
\end{bmatrix}
\]

So \( V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \) and therefore \( V \) is a subspace of \( \mathbb{R}^2 \) by Theorem 1.
Determining Subspaces: Example

Example

Is \( H = \left\{ \begin{bmatrix} a + 2b \\ a + 1 \\ a \end{bmatrix} : a \text{ and } b \text{ are real} \right\} \) a subspace of \( \mathbb{R}^3 \)?

Why or why not?

Solution: \( 0 \) is not in \( H \) since \( a = b = 0 \) or any other combination of values for \( a \) and \( b \) does not produce the zero vector. So property _____ fails to hold and therefore \( H \) is not a subspace of \( \mathbb{R}^3 \).
Example

Is the set $H$ of all matrices of the form
\[
\begin{bmatrix}
2a & b \\
3a + b & 3b
\end{bmatrix}
\]
a subspace of $M_{2\times2}$? Explain.

Solution: Since

\[
\begin{bmatrix}
2a & b \\
3a + b & 3b
\end{bmatrix} = \begin{bmatrix}
2a & 0 \\
3a & 0
\end{bmatrix} + \begin{bmatrix}
0 & b \\
b & 3b
\end{bmatrix}
\]

\[
= a \begin{bmatrix}
2 \\
3
\end{bmatrix} + b \begin{bmatrix}
0 \\
1
\end{bmatrix}.
\]

Therefore $H = \text{Span} \left\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right\}$ and so $H$ is a subspace of $M_{2\times2}$. 

Jiwen He, University of Houston
Spanning Sets

Theorem (1.5)

The span of any subset $S$ of a vector space $V$ is a subspace of $V$. Moreover, any subspace of $V$ that contains $S$ must also contain the span of $S$.

Definition

The **subspace spanned** (or **subspace generated**) by a nonempty set $S$ of vectors in $V$ is the set of all linear combinations of vectors from $S$:

$$< S > = \text{span}(S) = \{c_1v_1 + \cdots + c_nv_n \mid c_i \in F, v_i \in S\}$$

When $S = \{v_1, \cdots, v_n\}$ is a finite set, we use the notation $< v_1, \cdots, v_n >$ or $\text{span}(v_1, \cdots, v_n)$. A set $S$ of vectors in $V$ is said to **span** $V$, or **generate** $V$, if $V = \text{span}(S)$. 