Math 4377/6308 Advanced Linear Algebra
1.5 Linear Dependence and Linear Independence

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1.5 Linear Dependence and Linear Independence

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Linear Independence: Definition

**Linear Independence**
A set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in a vector space \( V \) is said to be **linearly independent** if the vector equation

\[
x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0
\]

has only the trivial solution \((x_1 = \cdots = x_p = 0)\).

**Linear Dependence**
The set \( \{v_1, v_2, \ldots, v_p\} \) is said to be **linearly dependent** if there exists weights \( c_1, \ldots, c_p, \) not all 0, such that

\[
c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0.
\]

↑
linear dependence relation
(when weights are not all zero)
Linear Independence and Homogeneous System

Example

A homogeneous system such as

\[
\begin{bmatrix}
1 & 2 & -3 \\
3 & 5 & 9 \\
5 & 9 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

can be viewed as a vector equation

\[
x_1 \begin{bmatrix}
1 \\
3 \\
5
\end{bmatrix} + x_2 \begin{bmatrix}
2 \\
5 \\
9
\end{bmatrix} + x_3 \begin{bmatrix}
-3 \\
9 \\
3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
\]

The vector equation has the trivial solution \((x_1 = 0, x_2 = 0, x_3 = 0)\), but is this the only solution?
Let \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} \).

a. Determine if \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is linearly independent.

b. If possible, find a linear dependence relation among \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \).

**Solution: (a)**

\[
x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Augmented matrix:

\[
\begin{bmatrix}
1 & 2 & -3 & 0 \\
3 & 5 & 9 & 0 \\
5 & 9 & 3 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & -3 & 0 \\
0 & -1 & 18 & 0 \\
0 & -1 & 18 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & -3 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\( x_3 \) is a free variable \( \Rightarrow \) there are nontrivial solutions.
Linear Independence: Example (cont.)

⇒ \{v_1, v_2, v_3\} is a linearly dependent set

(b) Reduced echelon form:

\[
\begin{bmatrix}
1 & 0 & 33 & 0 \\
0 & 1 & -18 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\rightarrow
\begin{align*}
x_1 &= \\
x_2 &= \\
x_3 &= 
\end{align*}
\]

Let \( x_3 = \) _____ (any nonzero number).

Then \( x_1 = \) _____ and \( x_2 = \) ____.

\[
\begin{bmatrix}
1 \\
3 \\
5 \\
\end{bmatrix} + \begin{bmatrix}
2 \\
5 \\
9 \\
\end{bmatrix} + \begin{bmatrix}
-3 \\
9 \\
3 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

or

\[
\begin{align*}
\text{---}v_1 + \text{---}v_2 + \text{---}v_3 &= 0 \\
\text{(one possible linear dependence relation)}
\end{align*}
\]
### Example (Linear Dependence Relation)

\[
\begin{bmatrix}
-33 \\
3 \\
5 \\
\end{bmatrix} + 18 \begin{bmatrix}
2 \\
5 \\
9 \\
\end{bmatrix} + 1 \begin{bmatrix}
-3 \\
9 \\
3 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

can be written as the matrix equation:

\[
\begin{bmatrix}
1 & 2 & -3 \\
3 & 5 & 9 \\
5 & 9 & 3 \\
\end{bmatrix} \begin{bmatrix}
-33 \\
18 \\
1 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}.
\]

Each linear dependence relation among the columns of \( A \) corresponds to a nontrivial solution to \( Ax = 0 \).

The columns of matrix \( A \) are linearly independent if and only if the equation \( Ax = 0 \) has only the trivial solution.
Sometimes we can determine linear independence of a set with minimal effort.

**Example (1. A Set of One Vector)**

Consider the set containing one nonzero vector: \( \{v_1\} \)

The only solution to \( x_1v_1 = 0 \) is \( x_1 = \ldots \).

So \( \{v_1\} \) is linearly independent when \( v_1 \neq 0 \).
Example (2. A Set of Two Vectors)

Let

\[ \mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \]

a. Determine if \( \{\mathbf{u}_1, \mathbf{u}_2\} \) is a linearly dependent set or a linearly independent set.

b. Determine if \( \{\mathbf{v}_1, \mathbf{v}_2\} \) is a linearly dependent set or a linearly independent set.

**Solution:** (a) Notice that \( \mathbf{u}_2 = \ldots \mathbf{u}_1 \). Therefore

\[ \ldots \mathbf{u}_1 + \ldots \mathbf{u}_2 = 0 \]

This means that \( \{\mathbf{u}_1, \mathbf{u}_2\} \) is a linearly \ldots set.
(b) Suppose

\[ cv_1 + dv_2 = 0. \]

Then \( v_1 = \frac{1}{c}v_2 \) if \( c \neq 0 \). But this is impossible since \( v_1 \) is ______ a multiple of \( v_2 \) which means \( c = \ldots \).

Similarly, \( v_2 = \frac{1}{d}v_1 \) if \( d \neq 0 \).

But this is impossible since \( v_2 \) is not a multiple of \( v_1 \) and so \( d = 0 \).

This means that \( \{v_1, v_2\} \) is a linearly ____________ set.
A set of two vectors is linearly dependent if at least one vector is a multiple of the other.

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.
Special Cases: 3. A Set Containing the 0 Vector

**Theorem**

A set of vectors $S = \{v_1, v_2, \ldots, v_p\}$ in $V$ containing the zero vector is linearly dependent.

**Proof:** Renumber the vectors so that $v_1 = \_\_\_\_\_\_\_\_$. Then

$$\_\_\_\_\_\_\_\_v_1 + \_\_\_\_\_\_\_\_v_2 + \cdots + \_\_\_\_\_\_\_\_v_p = 0$$

which shows that $S$ is linearly ______________.
1.5 Linear Independence

Special Cases: 4. A Set Containing Too Many Vectors

**Theorem**

*If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. I.e. any set \( \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \) is linearly dependent if \( p > n \).*

**Outline of Proof:**

\[
A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix} \text{ is } n \times p
\]

Suppose \( p > n \).

\[\Rightarrow \quad Ax = 0 \text{ has more variables than equations} \]

\[\Rightarrow \quad Ax = 0 \text{ has nontrivial solutions} \]

\[\Rightarrow \text{columns of } A \text{ are linearly dependent} \]
Examples

With the least amount of work possible, decide which of the following sets of vectors are linearly independent and give a reason for each answer.

a. \[
\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}
\]

b. Columns of
\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 & 0 \\
9 & 8 & 7 & 6 & 5 \\
4 & 3 & 2 & 1 & 8
\end{bmatrix}
\]
Special Cases: Examples (cont.)

\begin{align*}
\text{c.} & \quad \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\
\text{d.} & \quad \left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} \right\}
\end{align*}
Characterization of Linearly Dependent Sets

Example

Consider the set of vectors \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \} \) in \( \mathbb{R}^3 \) in the following diagram. Is the set linearly dependent? Explain.
Characterization of Linearly Dependent Sets

Theorem

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in $S$ is a linear combination of the others. In fact, if $S$ is linearly dependent, and $\mathbf{v}_1 \neq \mathbf{0}$, then some vector $\mathbf{v}_j$ ($j \geq 2$) is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$. 
Properties of Linearly Independent Sets

- The empty set is linearly independent.
- A set with a single nonzero vector is linearly independent.
- A set is linearly independent if the only representations of 0 as a linear combination of its vectors are trivial.
Properties of Linear Dependence and Linear Independence

**Theorem (1.6)**

Let $V$ be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If $S_1$ is linearly dependent, then $S_2$ is linearly dependent.

**Corollary**

Let $V$ be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If $S_2$ is linearly independent, then $S_1$ is linearly independent.

**Theorem (1.7)**

Let $S$ be a linearly independent subset of a vector space $V$, and let $v$ be a vector in $V$ that is not in $S$. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$. 