Math 4377/6308 Advanced Linear Algebra

1.5 Linear Dependence and Linear Independence

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/∼jiwenhe/math4377

1.5 Linear Dependence and Linear Independence

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Linear Independence: Definition

Linear Independence

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $(x_1 = \cdots = x_p = 0)$.

Linear Dpendence

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, \ldots, c_p$,not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

$\uparrow$

linear dependence relation

(when weights are not all zero)

Linear Independence and Homogeneous System

Example

A homogeneous system such as

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be viewed as a vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The vector equation has the trivial solution ($x_1 = 0$, $x_2 = 0$, $x_3 = 0$), but is this the *only solution*?

Linear Independence: Example

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

a. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution: (a)

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free variable $\Rightarrow$ there are nontrivial solutions.

Linear Independence: Example (cont.)

$$\Rightarrow \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \text{ is a linearly dependent set}$$

(b) Reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{array}{ll} x_1 & = \\ x_2 & = \\ x_3 & \end{array}$$

Let $x_3 =$ _____ (any nonzero number).

Then $x_1 =$ _____ and $x_2 =$ _____.

$$\_\_\_\_ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\_\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_2 + \_\_\_\_\mathbf{v}_3 = \mathbf{0}$$

(one possible linear dependence relation)

Linear Independence of Matrix Columns

Example (Linear Dependence Relation)

$$-33\begin{bmatrix}1\\3\\5\end{bmatrix}+18\begin{bmatrix}2\\5\\9\end{bmatrix}+1\begin{bmatrix}-3\\9\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

can be written as the matrix equation:

$$\begin{bmatrix}1&2&-3\\3&5&9\\5&9&3\end{bmatrix}\begin{bmatrix}-33\\18\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution to $A\mathbf{x}=\mathbf{0}$.

The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x}=\mathbf{0}$ has *only* the trivial solution.

Special Cases: 1. A Set of One Vector

Sometimes we can determine linear independence of a set with minimal effort.

Example (1. A Set of One Vector)

Consider the set containing one nonzero vector: $\{\mathbf{v}_1\}$

The only solution to $x_1\mathbf{v}_1 = 0$ is $x_1 =$ _____.

So $\{\mathbf{v}_1\}$ is linearly independent when $\mathbf{v}_1 \neq \mathbf{0}$.

Special Cases: 2. A Set of Two Vectors

Example (2. A Set of Two Vectors)

Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

a. Determine if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly dependent set or a linearly independent set.

b. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set or a linearly independent set.

Solution: (a) Notice that $\mathbf{u}_2 = \_\_\_\_\_\mathbf{u}_1$. Therefore

$$\_\_\_\_\_\mathbf{u}_1 + \_\_\_\_\_\mathbf{u}_2 = 0$$

This means that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly _______________ set.

Special Cases: 2. A Set of Two Vectors (cont.)

(b) Suppose

$$c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}.$$

Then $\mathbf{v}_1 = \text{———}\mathbf{v}_2$ if $c \neq 0$. But this is impossible since $\mathbf{v}_1$ is ______ a multiple of $\mathbf{v}_2$ which means $c =$ _____.

Similarly, $\mathbf{v}_2 = \text{———}\mathbf{v}_1$ if $d \neq 0$.

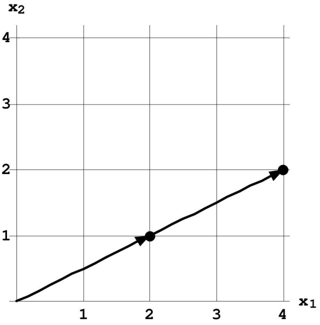
But this is impossible since $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ and so $d = 0$.

This means that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly ________________ set.

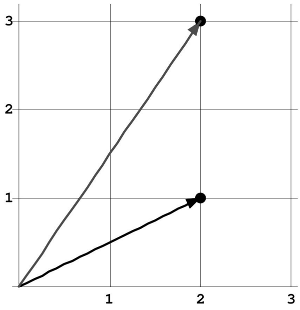
Special Cases: 2. A Set of Two Vectors (cont.)

A set of two vectors is linearly dependent if at least one vector is a multiple of the other.

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.



linearly ________________



linearly ________________

Special Cases: 3. A Set Containing the $\mathbf{0}$ Vector

Theorem

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in V containing the zero vector is linearly dependent.

Proof: Renumber the vectors so that $\mathbf{v}_1 =$ ____. Then

$$\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\mathbf{v}_p = \mathbf{0}$$

which shows that S is linearly ________________.

Special Cases: 4. A Set Containing Too Many Vectors

Theorem

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. I.e. any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ *in* $\mathbf{R}^n$ *is linearly dependent if* $p > n$*.*

Outline of Proof:

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p \end{bmatrix} \text{ is } n \times p$$

Suppose $p > n$.

$$\Longrightarrow A\mathbf{x} = \mathbf{0} \text{ has more variables than equations}$$

$$\Longrightarrow A\mathbf{x} = \mathbf{0} \text{ has nontrivial solutions}$$

$$\Longrightarrow \text{columns of } A \text{ are linearly dependent}$$

Special Cases: Examples

Examples

With the least amount of work possible, decide which of the following sets of vectors are linearly independent and give a reason for each answer.

a. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

b. Columns of $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 9 & 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 & 8 \end{bmatrix}$

Special Cases: Examples (cont.)

Examples (cont.)

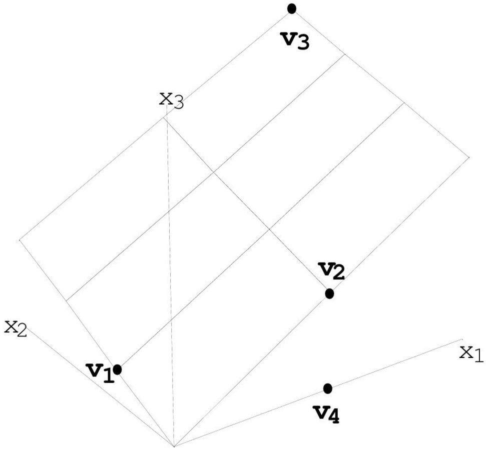
c. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} \right\}$

Characterization of Linearly Dependent Sets

Example

Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in $\mathbf{R}^3$ in the following diagram. Is the set linearly dependent? Explain



Characterization of Linearly Dependent Sets

Theorem

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent, and $\mathbf{v}_1 \neq \mathbf{0}$, then some vector $\mathbf{v}_j$ $(j \geq 2)$ is a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

Properties of Linearly Independent Sets

- The empty set is linearly independent
- A set with a single nonzero vector is linearly independent
- A set is linearly independent the only representations of 0 as a linear combination of its vectors are trivial

Properties of Linear Dependence and Linear Independence

Theorem (1.6)

Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

Corollary

Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

Theorem (1.7)

Let S be a linearly independent subset of a vector space V, and let $\mathbf{v}$ be a vector in V that is not in S. Then $S \cup \{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} \in \mathbf{span}(S)$.