Math 4377/6308 Advanced Linear Algebra

1.6 Bases and Dimension

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/∼jiwenhe/math4377

1.6 Bases and Dimension

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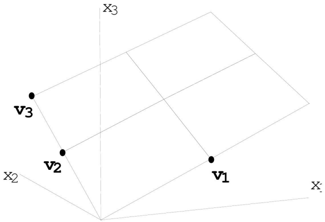
A Basis Set

Let H be the plane illustrated below. Which of the following are valid descriptions of H?

(a) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ (b) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$

(c) $H = \text{Span}\{\mathbf{v}_2, \mathbf{v}_3\}$ (d) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$



A *basis set* is an "efficient" spanning set containing no unnecessary vectors. In this case, we would consider the linearly independent sets $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_3\}$ to both be examples of basis sets or bases (plural for basis) for H.

A Basis Set: Definition

Definition

A basis β for a vector space V is a linearly independent subset of V that generates V. The vectors of β form a basis for V.

A Basis Set of Subspace

Let H be a subspace of a vector space V. An indexed set of vectors $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a basis for H if

i. β is a linearly independent set, and

ii. $H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$.

Example

Since $\mathrm{span}(\emptyset) = \{\mathbf{0}\}$ and $\emptyset$ is linearly independent, $\emptyset$ is a basis for the zero vector space.

A Basis Set: Examples

Example

Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Show that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbb{R}^3$.

Solutions:

$$\text{Let} \quad A = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since A has 3 pivots,

- the columns of A are linearly ______________________, by the IMT,
- and the columns of A _____________ _________ by IMT;
- therefore, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbb{R}^3$.

The basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is called a **standard basis** for F^n:
$\mathbf{e}_1 = (1, 0, \cdots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \cdots, 0)$, $\cdots$, $\mathbf{e}_n = (0, \cdots, 0, 1)$.

A Basis Set: Examples

Example

Let $S = \{1,\ x,\ x^2,\ \ldots, x^n\}$. Show that S is a basis for $\mathbf{P}_n$.

Solution: Any polynomial in $\mathbf{P}_n$ is in span of S.To show that S is linearly independent, assume

$$c_0 \cdot 1 + c_1 \cdot x + \cdots + c_n \cdot x^n = \mathbf{0}.$$

Then $c_0 = c_1 = \cdots = c_n = 0$. Hence S is a basis for $\mathbf{P}_n$.

The basis $\{1, x, x^2, \cdots, x^n\}$ is called the **standard basis** for $\mathbf{P}_n(F)$.

A Basis Set: Example

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a **basis** for $\mathbb{R}^3$?

Solution: Let $A = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. By row reduction,

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

and since there are 3 pivots, the columns of A are linearly independent and they span $\mathbb{R}^3$ by the IMT. Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a **basis** for $\mathbb{R}^3$.

A Basis Set: Example

Example

Explain why each of the following sets is **not** a basis for $\mathbb{R}^3$.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} \right\}$

Example

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

The Spanning Set Theorem

A basis can be constructed from a spanning set of vectors by discarding vectors which are linear combinations of preceding vectors in the indexed set.

Example

Suppose $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

Solution: If $\mathbf{x}$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\left(\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2\right)$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\_\_\_\_\_\_\_\mathbf{v}_2$$

Therefore,

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

The Spanning Set Theorem

Theorem (The Spanning Set Theorem)

Let

$$S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$$

be a set in V and let

$$H = \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}.$$

a. *If one of the vectors in S - say $\mathbf{v}_k$ - is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.*

b. *If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.*

Bases for Spanning Set: Theorem and Examples

Example

Find a basis for $H = \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$, where

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4] = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Solution: Row reduce:

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4] \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4]$$

Bases for Spanning Set: Theorem and Examples (cont.)

Note that

$$\mathbf{b}_2 = \_\_\_\_\mathbf{b}_1 \qquad \text{and} \qquad \mathbf{a}_2 = \_\_\_\_\mathbf{a}_1$$

$$\mathbf{b}_4 = 4\mathbf{b}_1 + 5\mathbf{b}_3 \qquad \text{and} \qquad \mathbf{a}_4 = 4\mathbf{a}_1 + 5\mathbf{a}_3$$

$\mathbf{b}_1$ and $\mathbf{b}_3$ are not multiples of each other

$\mathbf{a}_1$ and $\mathbf{a}_3$ are not multiples of each other

Elementary row operations on a matrix do not affect the linear dependence relations among the columns of the matrix.

Therefore

$$\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$$

and $\{\mathbf{a}_1, \mathbf{a}_3\}$ is a basis for H.

Bases for Spanning Set: Theorem and Example

Theorem

The pivot columns of a matrix $A = [\mathbf{a}_1, \cdots, \mathbf{a}_2]$ *form a basis for* $\mathrm{span}(\mathbf{a}_1, \cdots, \mathbf{a}_n)$.

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$. Find a basis for $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution: Let

$$A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -3 & 6 & 9 \end{bmatrix}$$

Bases for Spanning Set: Theorem and Example (cont.)

By row reduction, $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore a basis

for Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is $\left\{ \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \end{bmatrix} \right\}$.

Properties of Bases

Theorem (1.8)

Let V be a vector space and $\beta = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a subset of V. Then β is a basis for V if and only if each $\mathbf{v} \in V$ can be uniquely expressed as a linear combination of vectors of β:

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$$

for unique scalars $a_1, \cdots, a_n$.

Theorem (1.9)

If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.

The Replacement Theorem

Theorem (1.10 The Replacement Theorem)

Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.

Corollary 0

If a vector space V has a basis $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Proof: Suppose $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is a set of p vectors in V where $p > n$. If S is a linearly independent subset of V, the Replacement Theorem implies that $p \leq n$, a contradiction. Therefore $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ are linearly dependent.

The Replacement Theorem (cont.)

Corollary 1

Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.

Proof: Suppose β_1 is a basis for V consisting of exactly n vectors. Now suppose β_2 is any other basis for V. By the definition of a basis, we know that β_1 and β_2 are both linearly independent sets.

By Corollary 0, if β_1 has more vectors than β_2, then ______ is a linearly dependent set (which cannot be the case).

Again by Corollary 0, if β_2 has more vectors than β_1, then ______ is a linearly dependent set (which cannot be the case).

Therefore β_2 has exactly n vectors also. ■

The Dimension of a Vector Space: Definition

Dimension of a Vector Space

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be 0. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Corollary 2

Let V be a vector space with dimension n.

(a) Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.

(b) Any linearly independent subset of V that contains exactly n vectors is a basis for V.

(c) Every linearly independent subset of V can be extended to a basis for V.

The Dimension of a Vector Space: Examples

Example

The standard basis for $\mathbf{P}_3$ is $\{\qquad\qquad\qquad\qquad\qquad\qquad\}$. So $\dim$ $\mathbf{P}_3 =$ ______.

$\boxed{\text{In general, } \dim \mathbf{P}_n = n + 1.}$

Example

The standard basis for $\mathbb{R}^n$ is $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of I_n. So, for example, dim $\mathbb{R}^3 = 3$.

The Dimension of a Vector Space: Examples (cont.)

Example

Find a basis and the dimension of the subspace

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a, b, c, d \text{ are real} \right\}.$$

Solution: Since

$$\begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The Dimension of a Vector Space: Example (cont.)

$$W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

$$\text{where } \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

- Note that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, so by the Spanning Set Theorem, we may discard $\mathbf{v}_3$.
- $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for W. Also, dim $W =$____.

Dimensions of Subspaces of R^3

Example (Dimensions of subspaces of R^3)

1. *0-dimensional subspace* contains only the zero vector $\mathbf{0} = (0, 0, 0)$.
2. *1-dimensional subspaces.* Span$\{\mathbf{v}\}$ where $\mathbf{v} \neq \mathbf{0}$ is in $\mathbb{R}^3$.
3. These subspaces are ______________ through the origin.
4. *2-dimensional subspaces.* Span$\{\mathbf{u}, \mathbf{v}\}$ where $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^3$ and are not multiples of each other.
5. These subspaces are ______________ through the origin.
6. *3-dimensional subspaces.* Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent vectors in $\mathbb{R}^3$. This subspace is $\mathbb{R}^3$ itself because the columns of $A = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}]$ span $\mathbb{R}^3$ according to the IMT.

Dimensions of Subspaces: Theorem

Theorem (1.11)

Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and $\dim(W) \leq \dim(V)$*. Moreover, if* $\dim(W) = \dim(V)$*, then* $V = W$*.*

Corollary

If W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V.

Dimensions of Subspaces: Example

Example

Let $H = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Then H is a subspace of $\mathbb{R}^3$ and $\dim H < \dim \mathbb{R}^3$. We could expand the spanning set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ to $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ for a basis of $\mathbb{R}^3$.