Math 4377/6308 Advanced Linear Algebra

2.1 Linear Transformations, Null Spaces and Ranges

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2.1 Linear Transformations, Null Spaces and Ranges

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Linear Transformations

Definition

We call a function $T : V \to W$ a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have

(a) $T(x + y) = T(x) + T(y)$ and

(b) $T(cx) = cT(x)$

1. If T is linear, then $T(0) = 0$.
2. T is linear $\Leftrightarrow$ $T(cx + y) = cT(x) + T(y)$, $\forall x, y \in V$, $c \in F$.
3. If T is linear, then $T(x - y) = T(x) - T(y)$, $\forall x, y \in V$.
4. T is linear $\Leftrightarrow$ for $x_1, \cdots, x_n \in V$ and $a_1, \cdots, a_n \in F$,

$$T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i).$$

Special Linear Transformations

1. The identity transformation $I_V : V \to V : I_V(x) = x, \forall x \in V$.
2. The zero transformation $T_0 : V \to W : T_0(x) = 0, \forall x \in V$.

Matrix Transformation

Suppose A is $m \times n$. The matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m : T_A(\mathbf{x}) = A\mathbf{x}, \forall \in \mathbb{R}^n$. Matrix A is an object acting on $\mathbf{x}$ by multiplication to produce a new vector $A\mathbf{x}$.

Solving $A\mathbf{x} = \mathbf{b}$ amounts to finding all ____ in $\mathbf{R}^n$ which are transformed into vector $\mathbf{b}$ in $\mathbf{R}^m$ through multiplication by A.

Terminology

$\mathbf{R}^n$: **domain** of T $\mathbf{R}^m$: **codomain** of T

$T(\mathbf{x})$ in $\mathbf{R}^m$ is the **image** of $\mathbf{x}$ under the transformation T

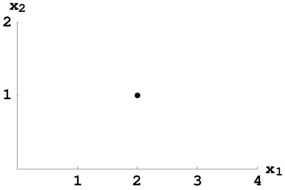
Set of all images $T(\mathbf{x})$ is the **range** of T

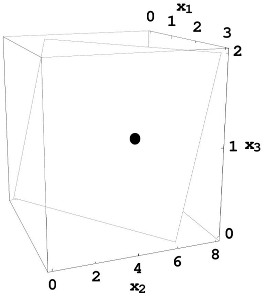
Matrix Transformations: Example

Example

Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Define $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

Then if $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$





Matrix Transformations: Example

Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Define a transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Find an $\mathbf{x}$ in $\mathbf{R}^3$ whose image under T is $\mathbf{b}$.

b. Is there more than one $\mathbf{x}$ under T whose image is $\mathbf{b}$. *(uniqueness problem)*

c. Determine if $\mathbf{c}$ is in the range of the transformation T. *(existence problem)*

Solution: (a) Solve ______=_____ for $\mathbf{x}$, or

$$\begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

Matrix Transformations: Example (cont.)

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ -5 & 10 & -15 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 2x_2 - 3x_3 + 2 \\ x_2 & \text{ is free} \\ x_3 & \text{ is free} \end{aligned}$$

Let $x_2 =$ ____ and $x_3 =$ ____. Then $x_1 =$ ____.

$$\text{So } \mathbf{x} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Matrix Transformations: Example (cont.)

(b) Is there an $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{b}$?

$$\text{Free variables exist} \\ \Downarrow \\ \text{There is more than one } \mathbf{x} \text{ for which } T(\mathbf{x}) = \mathbf{b}$$

(c) Is there an $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{c}$? This is another way of

asking if $A\mathbf{x} = \mathbf{c}$ is ______________.

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 3 \\ -5 & 10 & -15 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{c}$ is not in the ______________ of T.

Linear Transformations

If A is $m \times n$, then the transformation $T(\mathbf{x}) = A\mathbf{x}$ has the following properties:

$$T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_ + \_\_\_\_\_\_$$

and

$$T(c\mathbf{u}) = A(c\mathbf{u}) = \_\_\_\_\_A\mathbf{u} = \_\_\_\_\_T(\mathbf{u})$$

for all $\mathbf{u}$,$\mathbf{v}$ in $\mathbf{R}^n$ and all scalars c.

Every matrix transformation is a **linear** transformation.

Null Space and Range

Definition

For linear $T : V \to W$, the null space (or kernel) $N(T)$ of T is the set of all $x \in V$ such that $T(x) = 0$: $N(T) = \{x \in V : T(x) = 0\}$. The range (or image) $R(T)$ of T is the subset of W consisting of all images of vectors in V: $R(T) = \{T(x) : x \in V\}$.

Theorem (2.1)

For vector spaces V, W and linear $T : V \to W$, $N(T)$ and $R(T)$ are subspaces of V and W, respectively.

Theorem (2.2)

For vector spaces V, W and linear $T : V \to W$, if $\beta = \{v_1, \cdots, v_n\}$ is a basis for V, then

$$R(T) = \mathbf{span}(T(\beta)) = \mathbf{span}(\{T(v_1), \cdots, T(v_n)\}).$$

Null Space of a Matrix

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbf{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\} \qquad \text{(set notation)}$$

Theorem

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. Must verify properties a, b and c of the definition of a subspace.

Property (a) Show that $\mathbf{0}$ is in Nul A. Since _________, $\mathbf{0}$ is in

__________.

Null Space (cont.)

Property (b) If $\mathbf{u}$ and $\mathbf{v}$ are in Nul A, show that $\mathbf{u}+\mathbf{v}$ is in Nul A.
Since $\mathbf{u}$ and $\mathbf{v}$ are in Nul A,

$$\_\_\_\_\_\_\_\_\_\_ \text{ and } \_\_\_\_\_\_\_\_\_\_.$$

Therefore

$$A(\mathbf{u}+\mathbf{v}) = \_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_ = \_\_\_\_.$$

Property (c) If $\mathbf{u}$ is in Nul A and c is a scalar, show that $c\mathbf{u}$ in Nul A:

$$A(c\mathbf{u}) = \_\_\_A(\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

Since properties a, b and c hold, A is a subspace of $\mathbf{R}^n$.
Solving $A\mathbf{x} = \mathbf{0}$ yields an *explicit description of Nul A*.

Null Space: Example

Example

Find an explicit description of Nul A where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$$

Solution: Row reduce augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

Null Space: Example (cont.)

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \quad x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + \quad x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$\text{Nul } A = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$$

Null Space: Observations

Observations:

1. Spanning set of Nul A, found using the method in the last example, is automatically linearly independent:

$$c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$

$$c_1 = \_\_\_\_\_ \qquad c_2 = \_\_\_\_\_ \qquad c_3 = \_\_\_\_\_$$

2. If Nul A$\neq \{\mathbf{0}\}$, the the number of vectors in the spanning set for Nul A equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

Column Space of a Matrix

The **column space** of an $m \times n$ matrix A (Col A) is the set of all linear combinations of the columns of A.

If $A = [\mathbf{a}_1 \ \ldots \ \mathbf{a}_n]$, then

$$\boxed{\text{Col } A = \text{Span}\{\mathbf{a}_1, \ \ldots, \ \mathbf{a}_n\}}$$

Theorem

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

Why?

Recall that if $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ is a linear combination of the columns of A. Therefore

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbf{R}^n\}}$$

Column Space: Example

Example

Find a matrix A such that $W = Col\ A$ where

$$W = \left\{ \begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} : x, y \text{ in } \mathbf{R} \right\}.$$

Solution:

$$\begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} & \\ & \\ & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Column Space: Example (cont.)

Therefore

$$A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}.$$

The column space of an $m \times n$ matrix A is all of $\mathbf{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^m$.

The Contrast Between Nul A and Col A

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) The column space of A is a subspace of $\mathbf{R}^k$ where $k =$_____.

(b) The null space of A is a subspace of $\mathbf{R}^k$ where $k =$_____.

(c) Find a nonzero vector in Col A. (There are infinitely many possibilities.)

$$\_\_\_ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \_\_\_ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} + \_\_\_ \begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

The Contrast Between Nul A and Col A (cont.)

Example (cont.)

(d) Find a nonzero vector in Nul A. Solve $A\mathbf{x} = \mathbf{0}$ and pick one solution.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -2x_2 \\ x_2 & \text{ is free} \\ x_3 &= 0 \end{aligned} \implies \text{let } x_2 = \_\_\_ \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Contrast Between Nul A and Col A where A is $m \times n$

Null Spaces & Column Spaces: Examples

Example

Determine whether each of the following sets is a vector space or provide a counterexample.

(a) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x - y = 4 \right\}$

Solution: Since

$$\_\_\_\_ = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

is not in H, H is not a vector space.

Null Spaces & Column Spaces: Examples (cont.)

Example

(b) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{c} x - y = 0 \\ y + z = 0 \end{array} \right\}$

Solution: Rewrite

$$x - y = 0$$

$$y + z = 0$$

as

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $V =$Nul A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since Nul A is a subspace of $\mathbf{R}^2$, V is a vector space.

Null Spaces & Column Spaces: Examples (cont.)

Example

(c) $S = \left\{ \begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\}$

One *Solution:* Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$$S = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\};$$

therefore S is a vector space.

Null Spaces & Column Spaces: Examples (cont.)

Another Solution: Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$$S = \text{Col } A \qquad \text{where } A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 3 \end{bmatrix};$$

therefore S is a vector space, since a column space is a vector space.