Math 4377/6308 Advanced Linear Algebra

2.3 Composition of Linear Transformations

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2.3 Composition of Linear Transformations

- Compositions of Maps
- Basic Properties of Compositions
- Multiplication of Matrices

Composition of Linear Transformations

Theorem (2.9)

Let V, W, Z be vector spaces over a field F, and $T : V \to W$, $U : W \to Z$ be linear. Then $UT : V \to Z$ is linear.

Theorem (2.10)

Let V be a vector space and $T, U_1, U_2 \in \mathcal{L}(V)$. Then

(a) $T(U_1 + U_2) = TU_1 + TU_2$ *and* $(U_1 + U_2)T = U_1 T + U_2 T$.

(b) $T(U_1 U_2) = (TU_1)U_2$.

(c) $TI = IT = T$.

(d) $a(U_1 U_2) = (aU_1)U_2 = U_1(aU_2)$ *for all scalars* a.

Matrix Multiplication

Let $T : V \to W$, $U : W \to Z$ be linear, $\alpha = \{v_1, \cdots, v_n\}$, $\beta = \{w_1, \cdots, w_m\}$, $\gamma = \{z_1, \cdots, z_p\}$ ordered bases for V, W, Z, and $A = [U]_\beta^\gamma$, $B = [T]_\alpha^\beta$. Consider $[UT]_\alpha^\gamma$:

$$(UT)(v_j) = U(T(v_j)) = U\left(\sum_{k=1}^{m} B_{kj} w_k\right) = \sum_{k=1}^{m} B_{kj} U(w_k)$$

$$= \sum_{k=1}^{m} B_{kj} \left(\sum_{i=1}^{p} A_{ik} z_i\right) = \sum_{i=1}^{p} \left(\sum_{k=1}^{m} A_{ik} B_{kj}\right) z_i$$

Matrix Multiplication (cont.)

Definition

Let A, B be $m \times n$, $n \times p$ matrices. The product AB is the $m \times p$ matrix with

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}, \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq p.$$

Theorem (2.11)

Let V, W, Z be finite-dimensional vector spaces with ordered bases α, β, γ, and $T : V \to W$, $U : W \to Z$ be linear. Then

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

Matrix Multiplication (cont.)

Corollary

Let V be a finite-dimensional vector space with ordered basis β, and $T, U \in \mathcal{L}(V)$. Then $[UT]_\beta = [U]_\beta [T]_\beta$.

Definition

The Kronecker delta is defined by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The $n \times n$ identity matrix I_n is defined by $(I_n)_{ij} = \delta_{ij}$.

Matrix Notation

Matrix Notation

Two ways to denote $m \times n$ matrix A:

1. In terms of the *columns* of A:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

2. In terms of the *entries* of A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Main diagonal entries:________________

Matrix Addition: Theorem

Theorem (Addition)

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

a. $A+B=B+A$

b. $(A+B)+C=A+(B+C)$

c. $A+0=A$

d. $r(A+B)=rA+rB$

e. $(r+s)A=rA+sA$

f. $r(sA)=(rs)A$

Zero Matrix

$$0=\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

Matrix Multiplication

Matrix Multiplication

Multiplying B and $\mathbf{x}$ transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. In turn, if we multiply A and $B\mathbf{x}$, we transform $B\mathbf{x}$ into $A(B\mathbf{x})$. So $A(B\mathbf{x})$ is the composition of two mappings.

Define the product AB so that $A(B\mathbf{x}) = (AB)\mathbf{x}$.

Matrix Multiplication: Definition

Suppose A is $m \times n$ and B is $n \times p$ where

$$B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_p] \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p$$

and

$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p) \\ &= A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \cdots + A(x_p\mathbf{b}_p) \\ &= x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_pA\mathbf{b}_p = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}. \end{aligned}$$

Matrix Multiplication: Definition (cont.)

Therefore,

$$A(B\mathbf{x}) = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]\,\mathbf{x}.$$

and by defining

$$\boxed{AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]}$$

we have $A(B\mathbf{x}) = (AB)\,\mathbf{x}$.

Note that $A\mathbf{b}_1$ is a linear combination of the columns of A, $A\mathbf{b}_2$ is a linear combination of the columns of A, *etc*.

Each column of AB is a linear combination of the columns of A using weights from the corresponding columns of B.

Matrix Multiplication: Example

Example

Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

Solution:

$$A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix}, \qquad A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

$$\Longrightarrow AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Matrix Multiplication: Example

Example

If A is 4×3 and B is 3×2, then what are the sizes of AB and BA?

Solution:

$$AB = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \\ & \end{bmatrix}$$

$$BA \text{ would be } \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

which is ________________.

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

Row-Column Rule for Computing AB (alternate method)

The definition $\boxed{AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]}$ is good for theoretical work. When A and B have small sizes, the following method is more efficient when working by hand.

Row-Column Rule for Computing AB

If AB is defined, let $(AB)_{ij}$ denote the entry in the ith row and jth column of AB. Then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj},$$

i.e.,

$$\begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} = \begin{bmatrix} & & \\ & (AB)_{ij} & \\ & & \end{bmatrix}$$

Row-Column Rule for Computing AB: Example

Example

$A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined.

Solution: Since A is 2×3 and B is 3×2, then AB is defined and AB is _____$\times$_____.

$$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & \mathbf{-3} \\ 0 & \mathbf{1} \\ 4 & \mathbf{-7} \end{bmatrix} = \begin{bmatrix} 28 & \mathbf{-45} \\ \blacksquare & \blacksquare \end{bmatrix}$$

Row-Column Rule for Computing AB: Example (cont.)

$$\begin{bmatrix} 2 & 3 & 6 \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ \mathbf{2} & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 6 \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 2 & \mathbf{-3} \\ 0 & \mathbf{1} \\ 4 & \mathbf{-7} \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ 2 & \mathbf{-4} \end{bmatrix}$$

So $AB = \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$.

Matrix Multiplication: Theorem

Theorem (Multiplication)

Let A be $m \times n$ and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ *(associative law of multiplication)*

b. $A(B + C) = AB + AC$ *(left - distributive law)*

c. $(B + C)A = BA + CA$ *(right-distributive law)*

d. $r(AB) = (rA)B = A(rB)$ *for any scalar r*

e. $I_m A = A = A I_n$ *(identity for matrix multiplication)*

Matrix Power

Powers of A

$$A^k = \underbrace{A \cdots A}_{k}$$

Example

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 21 & 8 \end{bmatrix}$$

Properties of Matrix Multiplication

Theorem (2.12)

Let A be $m \times n$ matrix, B, C be $n \times p$ matrices, and D, E be $q \times m$ matrices. Then

(a) $A(B + C) = AB + AC$ *and* $(D + E)A = DA + EA$.

(b) $a(AB) = (aA)B = A(aB)$ *for any scalar* a.

(c) $I_m A = A = A I_n$.

(d) *If V is an n-dimensional vector space with ordered basis β, then* $[I_V]_\beta = I_n$.

Properties of Matrix Multiplication (cont.)

Corollary

Let A be $m \times n$ matrix, $B_1, \cdots, B_k$ be $n \times p$ matrices, $C_1, \cdots, C_k$ be $q \times m$ matrices, and $a_1, \cdots, a_k$ be scalars. Then

$$A\left(\sum_{i=1}^{k} a_i B_i\right) = \sum_{i=1}^{k} a_i A B_i$$

and

$$\left(\sum_{i=1}^{k} a_i C_i\right) A = \sum_{i=1}^{k} a_i C_i A$$

Properties of Matrix Multiplication (cont.)

Theorem (2.13)

Let A be $m \times n$ matrix and B be $n \times p$ matrix, and u_j, v_j the jth columns of AB, B. Then

(a) $u_j = Av_j$.

(b) $v_j = Be_j$.

Theorem (2.14)

Let V, W be finite-dimensional vector spaces with ordered bases β, γ, and $T : V \to W$ be linear. Then for $u \in V$:

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$$

Left-multiplication Transformations

Definition

Let A be $m \times n$ matrix. The left-multiplication transformation L_A is the mapping $L_A : F^n \to F^m$ defined by $L_A(x) = Ax$ for each column vector $x \in F^n$.

Left-multiplication Transformations (cont.)

Theorem (2.15)

Let A be $m \times n$ matrix, then $L_A : F^n \to F^m$ is linear, and if B is $m \times n$ matrix and β, γ are standard ordered bases for F^n, F^m, then:

(a) $[L_A]_\beta^\gamma = A$.

(b) $L_A = L_B$ *if and only if* $A = B$.

(c) $L_{A+B} = L_A + L_B$ *and* $L_{aA} = aL_A$ *for all* $a \in F$.

(d) *For linear* $T : F^n \to F^m$,*there exists a unique* $m \times n$ *matrix* C *such that* $T = L_C$, *and* $C = [T]_\beta^\gamma$.

(e) *If* E *is an* $n \times p$ *matrix, then* $L_{AE} = L_A L_E$.

(f) *If* $m = n$, *then* $L_{I_n} = I_{F^n}$.

Theorem (2.16)

Let A, B, C be matrices such that $A(BC)$ is defined. Then $(AB)C$ is also defined and $A(BC) = (AB)C$.