# Math 4377/6308 Advanced Linear Algebra 2.3 Composition of Linear Transformations

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# 2.3 Composition of Linear Transformations

- Compositions of Maps
- Basic Properties of Compositions
- Multiplication of Matrices



# Composition of Linear Transformations

## Theorem (2.9)

Let V, W, Z be vector spaces over a field F, and T :  $V \rightarrow W$ , U :  $W \rightarrow Z$  be linear. Then UT :  $V \rightarrow Z$  is linear.

## Theorem (2.10)

Let V be a vector space and T,  $U_1$ ,  $U_2 \in \mathcal{L}(V)$ . Then

(a) 
$$T(U_1 + U_2) = TU_1 + TU_2$$
 and  $(U_1 + U_2)T = U_1T + U_2T$ .

(b) 
$$T(U_1U_2) = (TU_1)U_2$$
.

$$(c) TI = IT = T.$$

(d)  $a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$  for all scalars a.

## Matrix Multiplication

Let  $T: V \to W$ ,  $U: W \to Z$  be linear,  $\alpha = \{v_1, \dots, v_n\}$ ,  $\beta = \{w_1, \dots, w_m\}$ ,  $\gamma = \{z_1, \dots, z_p\}$  ordered bases for V, W, Z, and  $A = [U]_{\beta}^{\gamma}$ ,  $B = [T]_{\alpha}^{\beta}$ . Consider  $[UT]_{\alpha}^{\gamma}$ :

$$(UT)(v_j) = U(T(v_j)) = U\left(\sum_{k=1}^{m} B_{kj} w_k\right) = \sum_{k=1}^{m} B_{kj} U(w_k)$$
$$= \sum_{k=1}^{m} B_{kj} \left(\sum_{i=1}^{p} A_{ik} z_i\right) = \sum_{i=1}^{p} \left(\sum_{k=1}^{m} A_{ik} B_{kj}\right) z_i$$



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# Matrix Multiplication (cont.)

#### Definition

Let A, B be  $m \times n$ ,  $n \times p$  matrices. The product AB is the  $m \times p$  matrix with

2.3 Composition

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}, \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq p.$$

## Theorem (2.11)

Let V, W, Z be finite-dimensional vector spaces with ordered bases  $\alpha$ ,  $\beta$ ,  $\gamma$ , and T : V  $\rightarrow$  W, U : W  $\rightarrow$  Z be linear. Then

$$[UT]^{\gamma}_{lpha} = [U]^{\gamma}_{eta} [T]^{eta}_{lpha}$$



#### Corollary

Let V be a finite-dimensional vector space with ordered basis  $\beta$ , and T,  $U \in \mathcal{L}(V)$ . Then  $[UT]_{\beta} = [U]_{\beta}[T]_{\beta}$ .

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#### Definition

The Kronecker delta is defined by  $\delta_{ij} = 1$  if i = j and  $\delta_{ij} = 0$  if  $i \neq j$ . The  $n \times n$  identity matrix  $I_n$  is defined by  $(I_n)_{ij} = \delta_{ij}$ .



## Matrix Notation

#### Matrix Notation

Two ways to denote  $m \times n$  matrix A:

In terms of the columns of A:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

In terms of the *entries* of A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

#### Main diagonal entries:\_\_\_\_



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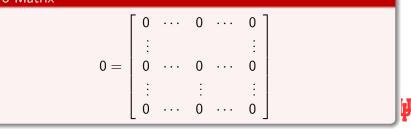
#### Theorem (Addition)

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

2.3 Composition

a. 
$$A + B = B + A$$
d.  $r(A + B) = rA + rB$ b.  $(A + B) + C = A + (B + C)$ e.  $(r + s)A = rA + sA$ c.  $A + 0 = A$ f.  $r(sA) = (rs)A$ 

Zero Matrix



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## Matrix Multiplication

#### Matrix Multiplication

Multiplying *B* and **x** transforms **x** into the vector *B***x**. In turn, if we multiply *A* and *B***x**, we transform *B***x** into  $A(B\mathbf{x})$ . So  $A(B\mathbf{x})$  is the composition of two mappings.

## Define the product AB so that $A(B\mathbf{x}) = (AB)\mathbf{x}$ .



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# Matrix Multiplication: Definition

Suppose A is  $m \times n$  and B is  $n \times p$  where

2.3 Composition

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

Then

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$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \dots + x_p\mathbf{b}_p$$
  
and  
$$A(B\mathbf{x}) = A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \dots + x_p\mathbf{b}_p)$$
  
$$= A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \dots + A(x_p\mathbf{b}_p)$$
  
$$x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \dots + x_pA\mathbf{b}_p = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_p] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

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Spring, 2015

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# Matrix Multiplication: Definition (cont.)

Therefore,

$$A(B\mathbf{x}) = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_p]\mathbf{x}.$$

and by defining

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

we have  $A(B\mathbf{x}) = (AB)\mathbf{x}$ .

Note that  $A\mathbf{b}_1$  is a linear combination of the columns of A,  $A\mathbf{b}_2$  is a linear combination of the columns of A, *etc*.

Each column of AB is a linear combination of the columns of A using weights from the corresponding columns of B.

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# Matrix Multiplication: Example

Example

Compute AB where 
$$A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$$
 and  $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$ .

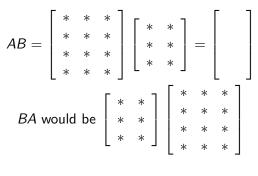
#### Solution:

## Matrix Multiplication: Example

#### Example

If A is  $4 \times 3$  and B is  $3 \times 2$ , then what are the sizes of AB and BA?

Solution:



which is \_\_\_\_\_

If A is  $m \times n$  and B is  $n \times p$ , then AB is  $m \times p$ .

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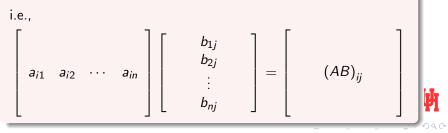
# Row-Column Rule for Computing AB (alternate method)

The definition  $AB = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_p]$  is good for theoretical work. When *A* and *B* have small sizes, the following method is more efficient when working by hand.

## Row-Column Rule for Computing AB

If AB is defined, let  $(AB)_{ij}$  denote the entry in the ith row and jth column of AB. Then

$$(AB)_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj},$$



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# 2.3 Composition Composition Addition Multiplication Row-Column Rule for Computing AB: Example

#### Example

$$A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}.$$
 Compute *AB*, if it is defined.

**Solution:** Since A is  $2 \times 3$  and B is  $3 \times 2$ , then AB is defined and AB is \_\_\_\_\_×\_\_\_.

$$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & \bullet \\ \bullet & \bullet \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & \mathbf{1} \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & -45 \\ \bullet & \bullet \end{bmatrix}$$

2.3 Composition

$$\begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ 2 & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 6 \\ -\mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 2 & -\mathbf{3} \\ 0 & \mathbf{1} \\ 4 & -\mathbf{7} \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ 2 & -\mathbf{4} \end{bmatrix}$$

So 
$$AB = \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$$
.

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## Matrix Multiplication: Theorem

#### Theorem (Multiplication)

Let A be  $m \times n$  and let B and C have sizes for which the indicated sums and products are defined.

- a. A(BC) = (AB)C (associative law of multiplication)
- b. A(B + C) = AB + AC (left distributive law)
- c. (B + C)A = BA + CA (right-distributive law)

d. 
$$r(AB) = (rA)B = A(rB)$$
  
for any scalar r

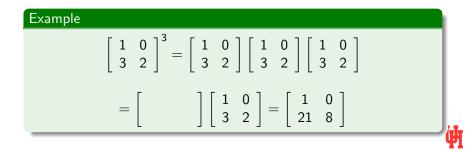
e.  $I_m A = A = A I_n$  (identity for matrix multiplication)



## Matrix Power

Powers of A

$$A^k = \underbrace{A \cdots A}_k$$



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# Properties of Matrix Multiplication

#### Theorem (2.12)

Let A be  $m \times n$  matrix, B, C be  $n \times p$  matrices, and D, E be  $q \times m$  matrices. Then

(a) 
$$A(B+C) = AB + AC$$
 and  $(D+E)A = DA + EA$ .

(b) 
$$a(AB) = (aA)B = A(aB)$$
 for any scalar a.

(c) 
$$I_m A = A = A I_n$$
.

(d) If V is an n-dimensional vector space with ordered basis  $\beta$ , then  $[I_V]_{\beta} = I_n$ .



Properties of Matrix Multiplication (cont.)

2.3 Composition

#### Corollary

Let A be  $m \times n$  matrix,  $B_1, \dots, B_k$  be  $n \times p$  matrices,  $C_1, \dots, C_k$  be  $q \times m$  matrices, and  $a_1, \dots, a_k$  be scalars. Then

$$A\left(\sum_{i=1}^k a_i B_i\right) = \sum_{i=1}^k a_i A B_i$$

and

$$\left(\sum_{i=1}^k a_i C_i\right) A = \sum_{i=1}^k a_i C_i A$$



Properties of Matrix Multiplication (cont.)

2.3 Composition

## Theorem (2.13)

Let A be  $m \times n$  matrix and B be  $n \times p$  matrix, and  $u_j$ ,  $v_j$  the jth columns of AB, B. Then

(a) 
$$u_j = Av_j$$
.  
(b)  $v_j = Be_j$ .

## Theorem (2.14)

Let V, W be finite-dimensional vector spaces with ordered bases  $\beta$ ,  $\gamma$ , and  $T : V \rightarrow W$  be linear. Then for  $u \in V$ :

$$[T(u)]_{\gamma} = [T]_{\beta}^{\gamma}[u]_{\beta}$$

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## Left-multiplication Transformations

#### Definition

Let A be  $m \times n$  matrix. The left-multiplication transformation  $L_A$  is the mapping  $L_A : F^n \to F^m$  defined by  $L_A(x) = Ax$  for each column vector  $x \in F^n$ .



## Theorem (2.15)

Let A be  $m \times n$  matrix, then  $L_A : F^n \to F^m$  is linear, and if B is  $m \times n$  matrix and  $\beta$ ,  $\gamma$  are standard ordered bases for  $F^n$ ,  $F^m$ , then:

#### Theorem (2.16)

Let A, B, C be matrices such that A(BC) is defined. Then (AB)C is also defined and A(BC) = (AB)C.



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