Math 4377/6308 Advanced Linear Algebra

2.4 Invertibility and Isomorphisms

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2.4 Invertibility and Isomorphisms

- Isomorphisms and Inverses
- Every finite dimensional vector space is isomorphic to coordinate space.

Inverse of Linear Transformation

Definition

Let V, W be vector spaces and $T : V \to W$ be linear. A function $U : W \to V$ is an inverse of T if $TU = I_W$ and $UT = I_V$. If T has an inverse, it is **invertible** and the inverse T^{-1} is unique.

For invertible T, U:

1. $(TU)^{-1} = U^{-1}T^{-1}$.
2. $(T^{-1})^{-1} = T$ (so T^{-1} is invertible)
3. If V, W have equal dimensions, linear $T : V \to W$ is invertible if and only if $\mathrm{rank}(T) = \dim(V)$.

Theorem (2.17)

For vector spaces V, W and linear and invertible $T : V \to W$, $T^{-1} : W \to V$ is linear.

The Inverse of a Matrix: Definition

The inverse of a real number a is denoted by a^{-1}. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1.$$

The Inverse of a Matrix

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A .

The Inverse of a Matrix: Facts

Fact

If A is invertible, then the inverse is unique.

Proof: Assume B and C are both inverses of A. Then

$$B = BI = B(\_\_\_\_\_) = (\_\_\_\_\_\_)\,\_\_\_\_\_\_ = I\,\_\_\_\_\_\_ = C.$$

So the inverse is unique since any two inverses coincide.■

Notation

The inverse of A is usually denoted by A^{-1}.

We have

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

Not all $n \times n$ matrices are invertible. A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.

The Inverse of a 2-by-2 Matrix

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$, *then* A *is invertible and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, *then* A *is not invertible.*

The Inverse of a Matrix: Solution of Linear System

Theorem

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof: Assume A is any invertible matrix and we wish to solve $A\mathbf{x} = \mathbf{b}$. Then

$$\_\_\_\_\_A\mathbf{x} = \_\_\_\_\_\mathbf{b} \qquad \text{and so}$$
$$I\mathbf{x} = \_\_\_\_\_\_ \text{ or } \mathbf{x} = \_\_\_\_\_\_.$$

Suppose $\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{w} = \mathbf{b}$ and

$$\_\_\_\_\_A\mathbf{w} = \_\_\_\_\_\mathbf{b} \qquad \text{which means} \qquad \mathbf{w} = A^{-1}\mathbf{b}.$$

So, $\mathbf{w} = A^{-1}\mathbf{b}$, which is in fact the same solution.

Solution of Linear System

Example

Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{array}{rcrcr} -7x_1 & + & 3x_2 & = & 2 \\ 5x_1 & - & 2x_2 & = & 1 \end{array}.$$

Solution: Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-15} \begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

The Inverse of a Matrix: Theorem

Theorem

Suppose A and B are invertible. Then the following results hold:

a. A^{-1} *is invertible and* $\left(A^{-1}\right)^{-1} = A$ *(i.e. A is the inverse of A^{-1}).*

b. AB *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$

c. A^T *is invertible and* $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

Partial proof of part b:

$$(AB)\left(B^{-1}A^{-1}\right) = A\left(\_\_\_\_\_\_\_\_\right)A^{-1}$$
$$= A\left(\_\_\_\_\_\_\_\_\_\_\right)A^{-1} = \_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_.$$

Similarly, one can show that $\left(B^{-1}A^{-1}\right)(AB) = I$.

Part b of Theorem can be generalized to three or more invertible matrices: $(ABC)^{-1} =$ ________

The Inverse of Elementary Matrix

Earlier, we saw a formula for finding the inverse of a 2×2 invertible matrix. How do we find the inverse of an invertible $n \times n$ matrix? To answer this question, we first look at **elementary** matrices.

Elementary Matrices

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

Example

Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

E_1, E_2, and E_3 are elementary matrices. Why?

Multiplication by Elementary Matrices

Observe the following products and describe how these products can be obtained by elementary row operations on A.

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operations on I_m.

The Inverses of Elementary Matrices: Example

Elementary matrices are *invertible* because row operations are *reversible*. To determine the inverse of an elementary matrix E, determine the elementary row operation needed to transform E back into I and apply this operation to I to find the inverse.

Example

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

The Inverses of Elementary Matrices: Example

Example

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2 (E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3 (E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Inverses of Elementary Matrices: Example (cont.)

Example (cont.)

So

$$\boxed{E_3 E_2 E_1 A = I_3}.$$

Then multiplying on the right by A^{-1}, we get

$$E_3 E_2 E_1 A\_\_\_\_\_\_ = I_3\_\_\_\_\_\_.$$

So

$$\boxed{E_3 E_2 E_1 I_3 = A^{-1}}$$

The Inverses of Elementary Matrices: Theorem

The elementary row operations that row reduce A to $\mathbf{I}_n$ are the same elementary row operations that transform $\mathbf{I}_n$ into $\mathbf{A}^{-1}$.

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}.

Algorithm for Finding $\mathbf{A}^{-1}$

Place A and I side-by-side to form an augmented matrix $[A \quad I]$. Then perform row operations on this matrix (which will produce identical operations on A and I). So by Theorem:

$$[A \quad I] \text{ will row reduce to } \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

or A is not invertible.

The Inverses of Matrix: Example

Example

Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$[A \quad I] = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

The Inverses of Matrix: Order

Order of multiplication is important!

Example

Suppose A,B,C, and D are invertible $n \times n$ matrices and
$A = B(D - I_n)C$.
Solve for D in terms of A, B, C and D.

Solution:

$$\_\_\_\_\_A\_\_\_\_\_ = \_\_\_\_\_B(D - I_n)C\_\_\_\_\_$$

$$D - I_n = B^{-1}AC^{-1}$$

$$D - I_n + \_\_\_\_ = B^{-1}AC^{-1} + \_\_\_\_$$

$$D = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Inverses

Lemma

For invertible and linear $T : V \to W$, V is finite-dimensional if and only if W is finite-dimensional. Then $\dim(V) = \dim(W)$.

Theorem (2.18)

Let V, W be finite-dimensional vector spaces with ordered bases β, γ, and $T : V \to W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible, and $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

Inverses (cont.)

Corollary 1

For finite-dimensional vector space V with ordered basis β and linear $T : V \to V$, T is invertible if and only if $[T]_\beta$ is invertible, and $[T^{-1}]_\beta = ([T_\beta])^{-1}$.

Corollary 2

An $n \times n$ matrix A is invertible if and only if L_A is invertible, and $(L_A)^{-1} = L_{A^{-1}}$.

Isomorphisms

Definition

Let V, W be vector spaces. V is **isomorphic** to W if there exists a linear transformation $T : V \to W$ that is invertible. Such a T is an isomorphism from V onto W.

Isomorphic

Informally, we say that vector space V is **isomorphic** to W if *every vector space calculation in* V *is accurately reproduced in* W, and vice versa.

Isomorphisms

Theorem (2.19)

For finite-dimensional vector spaces V and W, V is isomorphic to W if and only if $\dim(V) = \dim(W)$.

Corollary

A vector space V over F is isomorphic to F^n if and only if $\dim(V) = n$.

Isomorphisms

Theorem (2.20)

Let V, W be finite-dimensional vector spaces over F of dimensions n, m with ordered bases β, γ. Then the function $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(F)$, defined by $\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$, is an isomorphism.

Corollary

For finite-dimensional vector spaces V, W of dimensions n, m, $\mathcal{L}(V, W)$ is finite-dimensional of dimension mn.

The Standard Representation

Definition

Let β be an ordered basis for an n-dimensional vector space V over the field F. The standard representation of V with respect to β is the function $\phi_\beta : V \to F^n$ defined by $\phi_\beta(x) = [x]_\beta$ for each $x \in V$.

Theorem (2.21)

For any finite-dimensional vector space V with ordered basis β, ϕ_β is an isomorphism.

A set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in V is linearly independent if and only if $\left\{[\mathbf{u}_1]_\beta, [\mathbf{u}_2]_\beta, \ldots, [\mathbf{u}_p]_\beta\right\}$ is linearly independent in $\mathbf{F}^n$.

The Standard Representation: Example

Example

Use coordinate vectors to determine if $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a linearly independent set: $\mathbf{p}_1 = 1 - t$, $\mathbf{p}_2 = 2 - t + t^2$, $\mathbf{p}_3 = 2t + 3t^2$.

Solution: The standard basis set for $\mathbf{P}_2$ is $\beta = \{1, t, t^2\}$. So

$$[\mathbf{p}_1]_\beta = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \ [\mathbf{p}_2]_\beta = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \ [\mathbf{p}_3]_\beta = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

By the IMT, $\left\{[\mathbf{p}_1]_\beta, [\mathbf{p}_2]_\beta, [\mathbf{p}_3]_\beta\right\}$ is linearly __________________ and therefore $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly __________________.

The Standard Representation: Example

Coordinate vectors allow us to associate vector spaces with subspaces of other vectors spaces.

Example

Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

Let $H = \text{span}\{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_\beta$, if $\mathbf{x} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$.

Solution: (a) Find c_1 and c_2 such that

$$c_1 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$$

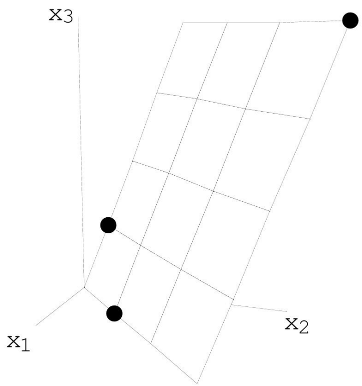
The Standard Representation: Example (cont.)

Corresponding augmented matrix:

$$\begin{bmatrix} 3 & 0 & 9 \\ 3 & 1 & 13 \\ 1 & 3 & 15 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $c_1 =$ ____ and $c_2 =$ _____ and so $[\mathbf{x}]_\beta = \begin{bmatrix} \quad \end{bmatrix}$.

The Standard Representation: Example (cont.)



$$\begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$$ in $\mathbf{R}^3$ is associated with the vector $$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$ in $\mathbf{R}^2$

H is isomorphic to $\mathbf{R}^2$