Math 4377/6308 Advanced Linear Algebra

2.5 Change of Bases & 2.6 Dual Spaces

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2.5 Change of bases

- Change of Coordinate Matrices.
- Similar Matrices.

The Change of Coordinate Matrix

Theorem (2.22)

Let β and β' be ordered bases for a finite-dimensional vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$. Then

(a) *Q is invertible.*

(b) *For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.*

The Change of Coordinate Matrix (cont.)

$Q = [I_V]_{\beta'}^{\beta}$ is called a **change of coordinate matrix**, and we say that Q changes β'-coordinates into β-coordinates.

Note that if Q changes from β' into β coordinates, then Q^{-1} changes from β into β' coordinates.

Linear Operators

A linear operator is a linear transformation from a vector space V into itself.

Theorem

Let T be a linear operator on a finite-dimensional vector space V with ordered bases β, β'. If Q is the change of coordinate matrix from β' into β-coordinates, then

$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q$$

Linear Operators (cont.)

Corollary

Let $A \in M_{n\times n}(F)$, and γ an ordered basis for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the $n \times n$ matrix with the vectors in γ as column vectors.

Definition

For A, $B \in M_{n\times n}(F)$, B is similar to A if the exists an invertible matrix Q such that $B = Q^{-1}AQ$.

2.6 Dual Spaces

- Dual Spaces and Dual Bases
- Transposes

Linear Functionals

A linear functional on a vector space V is a linear transformation from V into its field of scalars F.

Example

Let V be the continuous real-valued functions on $[0, 2\pi]$. For a fix $g \in V$, a linear functional $h : V \to R$ is given by

$$h(x) = \frac{1}{2\pi} \int_0^{2\pi} x(t)g(t)dt$$

Example

Let $V = M_{n\times n}(F)$, then $f : V \to F$ with $f(A) = \mathrm{tr}(A)$ is a linear functional.

Coordinate Functions

Example

Let $\beta = \{x_1, \cdots, x_n\}$ be a basis for a finite-dimensional vector space V. Define $f_i(x) = a_i$, where

$$[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

is the coordinate vector of x relative to β. Then f_i is a linear functional on V called the ith coordinate function with respect to the basis β. Note that $f_i(x_j) = \delta_{ij}$.

Dual Spaces

Definition

For a vector space V over F, the dual space of V is the vector space $V^* = \mathcal{L}(V, F)$.

Note that for finite-dimensional V,

$$\dim(V^*) = \dim(\mathcal{L}(V, F)) = \dim(V) \cdot \dim(F) = \dim(V)$$

so V and V^* are isomorphic. Also, the double dual V^{**} of V is the dual of V^*.

Dual Bases

Theorem (2.24)

Let $\beta = \{x_1, \cdots, x_n\}$ be an ordered basis for finite-dimensional vector space V, and let f_i be the ith coordinate function w.r.t. β, and $\beta^* = \{f_1, \cdots, f_n\}$. Then β^* is an ordered basis for V^* and for $f \in V^*$,

$$f = \sum_{i=1}^{n} f(x_i) f_i.$$

Definition

The ordered basis $\beta^* = \{f_1, \cdots, f_n\}$ of V^* that satisfies $f_i(x_j) = \delta_{ij}$ is called the dual basis of β.

Dual Bases (cont.)

Theorem (2.25)

Let V, W be finite-dimensional vector spaces over F with ordered bases β, γ. For any linear $T : V \to W$, the mapping $T^t : W^* \to V^*$ defined by $T^t(g) = gT$ for all $g \in W^*$ is linear with the property $[T^t]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$.

Double Dual Isomorphism

For a vector $x \in V$, define $\hat{x} : V^* \to F$ by $\hat{x}(f) = f(x)$ for every $f \in V^*$. Note that $\hat{x}$ is a linear functional on V^*, so $\hat{x} \in V^{**}$.

Lemma

For finite-dimensional vector space V and $x \in V$, if $\hat{x}(f) = 0$ for all $f \in V^*$, then $x = 0$.

Double Dual Isomorphism (cont.)

Theorem (2.26)

Let V be a finite-dimensional vector space, and define $\psi : V \to V^{**}$ by $\psi(x) = \hat{x}$. Then ψ is an isomorphism.

Corollary

For finite-dimensional V with dual space V^*, every ordered basis for V^* is the dual basis for some basis for V.