Math 4377/6308 Advanced Linear Algebra

3.2 The Rank of a Matrix and Matrix Inverses

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3.2 The Rank of a Matrix and Matrix Inverses

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The Rank of a Matrix

Definition (The Rank of a Matrix)

The rank of a matrix $A \in M_{m\times n}(F)$ is the rank of the linear transformation $L_A : F^n \to F^m$.

$$\text{rank}(A) = \text{rank}(L_A) = \dim(R(L_A))$$

An $n \times n$ matrix is invertible if and only if its rank is n.

The Rank of a Matrix

Theorem (3.3)

Let $T : V \to W$ be linear between finite-dimensional V, W with ordered bases β, γ. Then

$$\operatorname{rank}(T) = \operatorname{rank}([T]_\beta^\gamma).$$

$$\operatorname{rank}(T) = \operatorname{rank}(L_A), \operatorname{nullity}(T) = \operatorname{nullity}(L_A), \qquad \text{with } A = [T]_\beta^\gamma$$

Properties of the Rank of a Matrix

Theorem (3.4)

Let A be $m \times n$, and P, Q invertible of sizes $m \times m$, $n \times n$. Then

(a) $\text{rank}(AQ) = \text{rank}(A)$

(b) $\text{rank}(PA) = \text{rank}(A)$

(c) $\text{rank}(PAQ) = \text{rank}(A)$

(a) Note

$$R(L_{AQ}) = R(L_A L_Q) = L_A L_Q(F^n) = L_A(L_Q(F^n)) = L_A(F^n) = R(L_A).$$

Then

$$\text{rank}(AQ) = \dim(R(L_{AQ})) = \dim(R(L_A)) = \text{rank}(A).$$

Corollary

Elementary row/column operations are rank-preserving.

Determining the Rank of a Matrix

Theorem (3.5)

$\mathrm{rank}(A)$ *is the maximum number of linearly independent columns of A, that is, the dimension of the subspace generated by its columns.*

Note

$$R(L_A) = \mathrm{span}(\{L_A(e_1), \cdots, L_A(e_n)\}) = \mathrm{span}(\{a_1, \cdots, a_n\})$$

where $L_A(e_j) = Ae_j = a_j$, with a_j the jth column of A. Then

$$\mathrm{rank}(A) = \mathrm{rank}(L_A) = \dim(R(L_A)) = \dim(\mathrm{span}(\{a_1, \cdots, a_n\}))$$

Determining the Rank of a Matrix

Elementary row/column operations are rank-preserving.

Example (Row Reduction to Echelon Form)

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{rank}(A) = ?$$

Determining the Rank of a Matrix (cont.)

Theorem (3.6)

Let A be $m \times n$ with $\mathrm{rank}(A) = r$. Then $r \leq m$, $r \leq n$, and by finite number of elementary row/column operations A can be transformed into

$$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

where O_1, O_2, O_3 are zero matrices, that is, $D_{ii} = 1$ for $i \leq r$ and $D_{ij} = 0$ otherwise.

Elementary row/column operations are rank-preserving.

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{rank}(A) = r = 2$$

Determining the Rank of a Matrix (cont.)

Corollary 1

Let A be $m \times n$ of rank r. Then there exists invertible B, C of sizes $m \times m$, $n \times n$ such that

$$D = BAC = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

$$BAC = [B_{ij}?] \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} [C_{lk}?] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D$$

$$B = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad C = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

Determining the Rank of a Matrix (cont.)

Corollary 2

Let A be $m \times n$, then

(a) $\mathrm{rank}(A^t) = \mathrm{rank}(A)$

(b) $\mathrm{rank}(A)$ is the maximum number of linearly independent rows, that is, the dimension of the subspace generated by its rows.

(c) The rows and columns of A generate subspaces of the same dimension, namely $\mathrm{rank}(A)$

Corollary 3

Every invertible matrix is a product of elementary matrices.

Matrix Inverses as Products of Elementary Matrices

Example

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2 (E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3 (E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Inverses as Products of Elementary Matrices (cont.)

Example (cont.)

So

$$\boxed{E_3 E_2 E_1 A = I_3}.$$

Then multiplying on the right by A^{-1}, we get

$$E_3 E_2 E_1 A \_\_\_\_\_\_ = I_3 \_\_\_\_\_\_.$$

So

$$\boxed{E_3 E_2 E_1 I_3 = A^{-1}}$$

Rank of Matrix Products

Theorem (3.7)

Let $T : V \to W$ and $U : W \to Z$ be linear on finite-dimensional V, W, Z. Let A, B be matrices such that AB is defined. Then

(a) $\mathrm{rank}(UT) \leq \mathrm{rank}(U)$

(b) $\mathrm{rank}(UT) \leq \mathrm{rank}(T)$

(c) $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$

(d) $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$

The Inverse of a Matrix

Definition

Let A, B be $m \times n$, $m \times p$ matrices. The augmented matrix $(A|B)$ is the $m \times (n+p)$ matrix $(A\,B)$.

If A is invertible $n \times n$, then $(A|I_n)$ can be transformed into $(I_n|A^{-1})$ by finite number of elementary row operations.

If A is invertible $n \times n$ and $(A|I_n)$ is transformed into $(I_n|B)$ by finite number of elementary row operations, then $B = A^{-1}$.

If A is non-invertible $n \times n$, then any attempt to transform $(A|I_n)$ into $(I_n|B)$ produces a row whose first n entries are zero.

The Inverses of Matrix: Example

Example

Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$[A \quad I] = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$