# Math 4377/6308 Advanced Linear Algebra 3.4 Systems of Linear Equations – Computational Aspects

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# 3.4 Systems of Linear Equations – Computational Aspects

- Equivalent Systems
- Reduced Row Echelon Form
- Gaussian Elimination
- General Solutions
- Interpretation of the Reduced Row Echelon Form



# Equivalent Systems

### Definition

Two systems of linear equations are called **equivalent** if they have the same solution set.

### Theorem (3.13)

For  $m \times n$  linear system Ax = b and invertible  $m \times m$  matrix C, the system (CA)x = Cb is equivalent to Ax = b.



# Equivalent Systems

#### Corollary

For linear system Ax = b, if (A'|b') is obtained from (A|b) by a finite number of elementary row operations, then A'x = b' is equivalent to the original system.



# Reduced Row Echelon Form

### Definition

A matrix is in reduced row echelon form if:

- (a) Any row containing a nonzero entry precedes any row in which all the entries are zero
- (b) The first nonzero entry in each row is the only nonzero entry in its column
- (c) The first nonzero entry in each row is 1 and it occurs in a column right of the first nonzero entry in the preceding row.

#### Example

# Gaussian Elimination

#### Definition (Gaussian Elimination)

Reducing an augmented matrix to reduced row echelon form:

- In the forward pass, the matrix is transformed into upper triangular form where first nonzero entry of each row is 1, in a column to the right of the first nonzero entry of preceding rows.
- In the backward pass or back-substitution, the matrix is transformed into reduced row echelon form by making the first nonzero entry of each row the only nonzero entry of its column.



# Pivots

#### Important Terms

- **pivot position:** a position of a leading entry in an echelon form of the matrix.
- **pivot:** a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's.
- pivot column: a column that contains a pivot position.

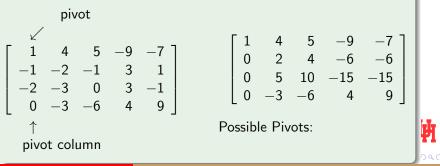


# Reduced Echelon Form: Examples

Example (Row reduce to echelon form and locate the pivots)

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

#### Solution

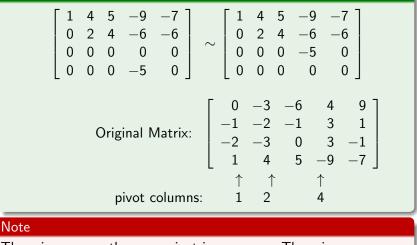


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8 / 19

# Reduced Echelon Form: Examples (cont.)

#### Example (Row reduce to echelon form (cont.))



There is no more than one pivot in any row. There is no more than one pivot in any column.

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# Reduced Echelon Form: Examples (cont.)

Example (Row reduce to echelon form and then to REF)

#### Solution:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$
$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

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# Reduced Echelon Form: Examples (cont.)

Example (Row reduce to echelon form and then to REF (cont.)) Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$
$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$
(echelon form)



# Reduced Echelon Form: Examples (cont.)

#### Example (Row reduce to echelon form and then to REF (cont.))

# **Final step to create the reduced echelon form:** Beginning with the rightmost leading entry, and working upwards to the left, create zeros above each leading entry and scale rows to transform each leading entry into 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Gaussian Elimination

# Theorem (3.14)

Gaussian elimination transforms any matrix into its reduced row echelon form.



# Solutions of Linear Systems

#### Important Terms

- **basic variable:** any variable that corresponds to a pivot column in the augmented matrix of a system.
- free variable: all nonbasic variables.

Example (Solutions of Linear Systems)

	Γ1	6	0	3	0	0	1
	0	0	1	3 -8 0	0	5	
	0	0	0	0	1	7	
$x_1$	$+6x_{2}$			$+3x_{4}$			= 0
_		_	<i>X</i> 3	-8x			= 5
			-			$X_5$	= 7

pivot columns: basic variables: free variables:

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# Solutions of Linear Systems (cont.)

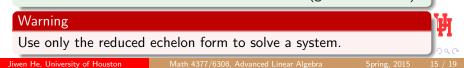
#### Final Step in Solving a Consistent Linear System

After the augmented matrix is in **reduced** echelon form and the system is written down as a set of equations, *Solve each equation* for the basic variable in terms of the free variables (if any) in the equation.

# Example (General Solutions of Linear Systems)

$$\begin{array}{ccccc} x_1 & +6x_2 & +3x_4 & = 0 \\ & x_3 & -8x_4 & = 5 \\ & & x_5 & = 7 \end{array} & \left\{ \begin{array}{c} x_1 = -6x_2 - 3x_1 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{array} \right.$$

# (general solution)



# General Solutions of Linear Systems

#### General Solution

The **general solution** of the system provides a parametric description of the solution set. (The free variables act as parameters.)

#### Example (General Solutions of Linear Systems (cont.))

$$x_1 = -6x_2 - 3x_4$$
  

$$x_2 \text{ is free}$$
  

$$x_3 = 5 + 8x_4$$
  

$$x_4 \text{ is free}$$
  

$$x_5 = 7$$

The above system has infinitely many solutions. Why?

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# General Solutions

#### Theorem (3.15)

Let Ax = b be a system of r nonzero equations in n unknowns. Suppose rank(A) = rank(A|b) and that (A|b) is in reduced row echelon form. Then

(a) 
$$rank(A) = r$$

(b) If the general solution is of the form

$$s = s_0 + t_1 u_1 + t_2 u_2 + \cdots + t_{n-r} u_{n-r}$$

then  $\{u_1, u_2, \dots, u_{n-r}\}$  is a basis for the solution set of the corresponding homogeneous system, and  $s_0$  is a solution to the original system.



# Interpretation of the Reduced Row Echelon Form

### Theorem (3.16)

Let A be an  $m \times n$  matrix of rank r > 0 and B the reduced row echelon form of A. Then

- (a) The number of nonzero rows in B is r.
- (b) For each  $i = 1, \dots, r$ , there is a column  $b_{j_i}$  of B s.t.  $b_{j_i} = e_i$
- (c) The columns of A numbered j<sub>1</sub>, ..., j<sub>r</sub> are linearly independent
- (d) For each  $k = 1, \dots, n$ , if column k of B is  $d_1e_1 + \dots + d_re_r$ then column k of A is  $d_1a_{j_1} + \dots + d_ra_{j_r}$



# Interpretation of the Reduced Row Echelon Form

Corollary

The reduced row echelon form of a matrix is unique.



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