Math 4377/6308 Advanced Linear Algebra

4.1 Determinants of Order 2

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4.1 Determinants of Order 2

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- Linearity
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Determinants of Order 2: Definition

Definition

For the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant of A, denoted $\det(A)$ or $|A|$, is the scalar $ad - bc$.

Determinants: Linearity

Theorem (4.1)

det : $M_{2\times 2}(F) \to F$ *is a linear function of each row of a* 2×2 *matrix when the other row is held fixed. That is, for* $u, v, w \in F^2$ *and* $k \in F$,

$$\det \begin{pmatrix} u + kv \\ w \end{pmatrix} = \det \begin{pmatrix} u \\ w \end{pmatrix} + k \det \begin{pmatrix} v \\ w \end{pmatrix}$$

$$\det \begin{pmatrix} w \\ u + kv \end{pmatrix} = \det \begin{pmatrix} w \\ u \end{pmatrix} + k \det \begin{pmatrix} w \\ v \end{pmatrix}$$

Determinants and Inverses

Theorem (4.2)

The determinant of $A \in M_{2\times 2}(F)$ is nonzero if and only if A is invertible. If A is invertible then

$$A^{-1} = \frac{1}{\det(A)} \det \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Determinant and Orientation of an Ordered Basis

The **orientation** of an ordered basis $\beta = \{u, v\}$ for $\mathbb{R}^2$ is defined by

$$\mathrm{Orient}\begin{pmatrix} u \\ v \end{pmatrix} == \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|}$$

Determinant and Left/Right-Handed Coordinate System

Note that $\mathrm{Orient}\begin{pmatrix} u \\ v \end{pmatrix} = \pm 1$, and $\mathrm{Orient}\begin{pmatrix} u \\ v \end{pmatrix} = 1$ if and only if $\{u, v\}$ forms a right-handed coordinate system (u can be rotated in a counterclockwise direction through an angle θ, with $0 < \theta < \pi$, to coincide with v).

Determinant and Area of a Parallelogram

The area of the parallelogram determined by u and v:

$$\mathrm{Area}\begin{pmatrix} u \\ v \end{pmatrix} = \mathrm{Orient}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \det\begin{pmatrix} u \\ v \end{pmatrix} = \left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|$$