Math 4377/6308 Advanced Linear Algebra

5.1 Eigenvalues and Eigenvectors

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Definition

A linear operator $T$ on a finite-dimensional vector space $V$ is diagonalizable if there is an ordered basis $\beta$ for $V$ such that $[T]_\beta$ is a diagonal matrix. A square matrix $A$ is diagonalizable if $L_A$ is diagonalizable.
Eigenvalues and Eigenvectors

Definition

Let $T$ be a linear operator on a vector space $V$. A nonzero vector $v \in V$ is an eigenvector of $T$ if there exists a scalar eigenvalue $\lambda$ corresponding to the eigenvector $v$ such that $T(v) = \lambda v$.

Let $A \in M_{n \times n}(F)$. A nonzero vector $v \in F^n$ is an eigenvector of $A$ if $v$ is an eigenvector of $L_A$; that is, if $Av = \lambda v$ for some scalar eigenvalue $\lambda$ of $A$ corresponding to the eigenvector $v$. 
Eigenvalues and Eigenvectors: Example

Example

Let \( A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \), \( u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), and \( v = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \). Examine the images of \( u \) and \( v \) under multiplication by \( A \).

Solution

\[
Au = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2u
\]

\( u \) is called an eigenvector of \( A \) since \( Au \) is a multiple of \( u \).

\[
Av = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \neq \lambda v
\]

\( v \) is not an eigenvector of \( A \) since \( Av \) is not a multiple of \( v \).

\( Au = -2u \), but \( Av \neq \lambda v \).
Show that 4 is an eigenvalue of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

**Solution:** Scalar 4 is an eigenvalue of $A$ if and only if $Ax = 4x$ has a nontrivial solution.

$$Ax - 4x = 0$$

$$(A - 4I)x = 0.$$ 

To solve $(A - 4I)x = 0$, we need to find $A - 4I$ first:

$$A - 4I = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}.$$
Now solve \((A-4I)x = 0\):

\[
\begin{bmatrix}
-4 & -2 & 0 \\
-4 & -2 & 0
\end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
x = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \\ \frac{1}{2} \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}.
\]

Each vector of the form \(x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\) is an eigenvector corresponding to the eigenvalue \(\lambda = 4\).

The set of all solutions to \((A-\lambda I)x = 0\) is called the \textbf{eigenspace} of \(A\) corresponding to \(\lambda\).
A linear operator $T$ on a finite-dimensional vector space $V$ is diagonalizable if and only if there exists an ordered basis $\beta$ for $V$ consisting of eigenvectors of $T$. If $T$ is diagonalizable, $\beta = \{v_1, \cdots, v_n\}$ is an ordered basis of eigenvectors of $T$, and $D = [T]_\beta$, then $D$ is a diagonal matrix and $D_{jj}$ is the eigenvalue corresponding to $v_j$ for $1 \leq j \leq n$. 
Diagonalization

To diagonalize a matrix or a linear operator is to find a basis of eigenvectors and the corresponding eigenvalues.
Theorem (5.2)

Let $A \in M_{n\times n}(F)$. Then a scalar $\lambda$ is an eigenvalue of $A$ if and only if $\det(A - \lambda I_n) = 0$. 
Characteristic Polynomial

**Definition**

Let $A \in M_{n \times n}(F)$. The polynomial $f(t) = \det(A - tl_n)$ is called the characteristic polynomial of $A$. 
Characteristic Polynomial

**Definition**

Let $T$ be a linear operator on an $n$-dimensional vector space $V$ with ordered basis $\beta$. We define the characteristic polynomial $f(t)$ of $T$ to be the characteristic polynomial of $A = [T]_{\beta}$:

$$f(t) = \det(A - tl_n).$$
Theorem (5.3)

Let $A \in M_{n \times n}(F)$.

(a) The characteristic polynomial of $A$ is a polynomial of degree $n$ with leading coefficient $(-1)^n$.

(b) $A$ has at most $n$ distinct eigenvalues.
Theorem (5.4)

Let $T$ be a linear operator on a vector space $V$, and let $\lambda$ be an eigenvalue of $T$. A vector $v \in V$ is an eigenvector of $T$ corresponding to $\lambda$ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$. 