Math 4377/6308 Advanced Linear Algebra

5.2 Diagonalizability

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5.2 Diagonalizability

- Diagonalizability
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Diagonalizability

Theorem (5.5)

Let T be a linear operator on a vector space V, and let $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of T. If $v_1, \cdots, v_k$ are the corresponding eigenvectors, then $\{v_1, \cdots, v_k\}$ is linearly independent.

Corollary

Let T be a linear operator on an n-dimensional vector space V. If T has n distinct eigenvalues, then T is diagonalizable.

Diagonalizability (cont.)

Definition

A polynomial $f(t)$ in $P(F)$ *splits over* F if there are scalars c, a_1, $\cdots$, a_n in F such that $f(t) = c(t - a_1)(t - a_2)\cdots(t - a_n)$.

Theorem (5.6)

The characteristic polynomial of any diagonalizable operator splits.

Multiplicity

Definition

Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The (algebraic) multiplicity of λ is the largest positive integer k for which $(t-\lambda)^k$ is a factor of $f(t)$.

Definition

Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Define $E_\lambda = \{x \in V : T(x) = \lambda x\} = N(T - I_V)$. The set E_λ is the eigenspace of T corresponding to the eigenvalue λ. The eigenspace of a square matrix A is the eigenspace of L_A.

Multiplicity (cont.)

Theorem (5.7)

Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of T having multiplicity m. Then $1 \leq \dim(E_\lambda) \leq m$.

Diagonalizability

Lemma

Let T be a linear operator, and let $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of T. For $i = 1, \cdots, k$, let $v_i \in E_{\lambda_i}$. If

$$v_1 + v_2 + \cdots + v_k = 0,$$

then $v_i = 0$ for all i.

Theorem (5.8)

Let T be a linear operator on a vector space V, and let $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of T. For $i = 1, \cdots, k$, let S_i be a finite linearly independent subset of the eigenspace E_{λ_i}. Then $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent subset of V.

Diagonalizability

Theorem (5.9)

Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits. Let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues of T. Then

(a) *T is diagonalizable if and only if the multiplicity of λ_i is equal to $\dim(E_{\lambda_i})$ for all i.*

(b) *If T is diagonalizable and β_i is an ordered basis for E_{λ_i}, for each i, then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors of T.*

Diagonalizability (cont.)

Test for Diagonalization

Let T be a linear operator on an n-dimensional vector space V. Then T is diagonalizable if and only if both of the following conditions hold.

- The characteristic polynomial of T splits.
- The multiplicity of each eigenvalue λ equals $n - \mathrm{rank}(T - \lambda I)$.

Direct Sums

Definition

The sum of the subspaces $W_1, \cdots, W_k$ of a vector space is the set

$$\sum_{i=1}^{k} W_i = \{v_1 + \cdots + v_k : v_i \in W_i \text{ for } 1 \le i \le k\}.$$

Definition

A vector space V is the direct sum of subspaces $W_1, \cdots, W_k$, denoted $V = W_1 \oplus \cdots \oplus W_k$, if

$$V = \sum_{i=1}^{k} W_i \text{ and } W_j \cap \sum_{i \neq j} W_i = \{0\} \text{ for each } j, 1 \le j \le k.$$

Direct Sums (cont.)

Theorem (5.10)

Let $W_1, \cdots, W_k$ be subspaces of finite-dimensional vector space V. The following are equivalent:

(a) $V = W_1 \oplus \cdots \oplus W_k$.

(b) $V = \sum_{i=1}^{k} W_i$ *and for any* $v_1, \cdots, v_k$ *s.t.* $v_i \in W_i$ $(1 \leq i \leq k)$, *if* $v_1 + \cdots + v_k = 0$, *then* $v_i = 0$ *for all* i.

(c) *Each* $v \in V$ *can be uniquely written as* $v = v_1 + \cdots + v_k$, *where* $v_i \in W_i$.

(d) *If* γ_i *is an ordered basis for* W_i $(1 \leq i \leq k)$, *then* $\gamma_1 \cup \cdots \cup \gamma_k$ *is an ordered basis for* V.

(e) *For each* $i = 1, \cdots, k$ *there exists an ordered basis* γ_i *for* W_i *such that* $\gamma_1 \cup \cdots \cup \gamma_k$ *is an ordered basis for* V.

Direct Sums (cont.)

Theorem (5.11)

A linear operator T on finite-dimensional vector space V is diagonalizable if and only if V is the direct sum of the eigenspaces of T.