Math 4377/6308 Advanced Linear Algebra

5.2 Diagonalizability

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5.2 Diagonalizability

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Diagonalizability

Theorem (5.5)

Let $T$ be a linear operator on a vector space $V$, and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of $T$. If $v_1, \ldots, v_k$ are the corresponding eigenvectors, then \{${v_1, \ldots, v_k}$\} is linearly independent.

Corollary

Let $T$ be a linear operator on an $n$-dimensional vector space $V$. If $T$ has $n$ distinct eigenvalues, then $T$ is diagonalizable.
Definition

A polynomial \( f(t) \) in \( P(F) \) \textit{splits over} \( F \) if there are scalars \( c, a_1, \ldots, a_n \) in \( F \) such that \( f(t) = c(t - a_1)(t - a_2) \cdots (t - a_n) \).

Theorem (5.6)

\textit{The characteristic polynomial of any diagonalizable operator splits.}
### Definition

Let $\lambda$ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The (algebraic) multiplicity of $\lambda$ is the largest positive integer $k$ for which $(t - \lambda)^k$ is a factor of $f(t)$.

### Definition

Let $T$ be a linear operator on a vector space $V$, and let $\lambda$ be an eigenvalue of $T$. Define $E_\lambda = \{x \in V : T(x) = \lambda x\} = N(T - I_V)$. The set $E_\lambda$ is the eigenspace of $T$ corresponding to the eigenvalue $\lambda$. The eigenspace of a square matrix $A$ is the eigenspace of $L_A$. 
Theorem (5.7)

Let $T$ be a linear operator on a finite-dimensional vector space $V$, and let $\lambda$ be an eigenvalue of $T$ having multiplicity $m$. Then

$$1 \leq \dim(E_\lambda) \leq m.$$
**Lemma**

Let $T$ be a linear operator, and let $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of $T$. For $i = 1, \cdots, k$, let $v_i \in E_{\lambda_i}$. If

$$v_1 + v_2 + \cdots + v_k = 0,$$

then $v_i = 0$ for all $i$.

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**Theorem (5.8)**

Let $T$ be a linear operator on a vector space $V$, and let $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of $T$. For $i = 1, \cdots, k$, let $S_i$ be a finite linearly independent subset of the eigenspace $E_{\lambda_i}$. Then $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent subset of $V$. 
Diagonalizability

Theorem (5.9)

Let $T$ be a linear operator on a finite-dimensional vector space $V$ such that the characteristic polynomial of $T$ splits. Let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues of $T$. Then

(a) $T$ is diagonalizable if and only if the multiplicity of $\lambda_i$ is equal to $\dim(E_{\lambda_i})$ for all $i$.

(b) If $T$ is diagonalizable and $\beta_i$ is an ordered basis for $E_{\lambda_i}$, for each $i$, then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for $V$ consisting of eigenvectors of $T$. 
Test for Diagonalization

Let $T$ be a linear operator on an $n$-dimensional vector space $V$. Then $T$ is diagonalizable if and only if both of the following conditions hold.

- The characteristic polynomial of $T$ splits.
- The multiplicity of each eigenvalue $\lambda$ equals $n - \text{rank}(T - \lambda I)$.
**Definition**

The sum of the subspaces \( W_1, \cdots, W_k \) of a vector space is the set

\[
\sum_{i=1}^{k} W_i = \{ v_1 + \cdots + v_k : v_i \in W_i \text{ for } 1 \leq i \leq k \}.
\]

**Definition**

A vector space \( V \) is the direct sum of subspaces \( W_1, \cdots, W_k \), denoted \( V = W_1 \oplus \cdots \oplus W_k \), if

\[
V = \sum_{i=1}^{k} W_i \text{ and } W_j \cap \sum_{i \neq j} W_i = \{0\} \text{ for each } j, 1 \leq j \leq k.
\]
Direct Sums (cont.)

Theorem (5.10)

Let $W_1, \ldots, W_k$ be subspaces of finite-dimensional vector space $V$. The following are equivalent:

(a) $V = W_1 \oplus \cdots \oplus W_k$.

(b) $V = \sum_{i=1}^{k} W_i$ and for any $v_1, \ldots, v_k$ s.t. $v_i \in W_i$ ($1 \leq i \leq k$), if $v_1 + \cdots + v_k = 0$, then $v_i = 0$ for all $i$.

(c) Each $v \in V$ can be uniquely written as $v = v_1 + \cdots + v_k$, where $v_i \in W_i$.

(d) If $\gamma_i$ is an ordered basis for $W_i$ ($1 \leq i \leq k$), then $\gamma_1 \cup \cdots \cup \gamma_k$ is an ordered basis for $V$.

(e) For each $i = 1, \ldots, k$ there exists an ordered basis $\gamma_i$ for $W_i$ such that $\gamma_1 \cup \cdots \cup \gamma_k$ is an ordered basis for $V$. 
Theorem (5.11)

A linear operator $T$ on finite-dimensional vector space $V$ is diagonalizable if and only if $V$ is the direct sum of the eigenspaces of $T$. 