4. Convex optimization problems

- optimization problem in standard form
- convex optimization problems
- quasiconvex optimization
- linear optimization
- quadratic optimization
- geometric programming
- generalized inequality constraints
- semidefinite programming
- vector optimization
Optimization problem in standard form

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, \quad i = 1, \ldots, m \)
\( h_i(x) = 0, \quad i = 1, \ldots, p \)

- \( x \in \mathbb{R}^n \) is the optimization variable
- \( f_0 : \mathbb{R}^n \to \mathbb{R} \) is the objective or cost function
- \( f_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, \ldots, m \), are the inequality constraint functions
- \( h_i : \mathbb{R}^n \to \mathbb{R} \) are the equality constraint functions

optimal value:

\[ p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0, \ i = 1, \ldots, m, \ h_i(x) = 0, \ i = 1, \ldots, p \} \]

- \( p^* = \infty \) if problem is infeasible (no \( x \) satisfies the constraints)
- \( p^* = -\infty \) if problem is unbounded below
Optimal and locally optimal points

$x$ is **feasible** if $x \in \text{dom} f_0$ and it satisfies the constraints

a feasible $x$ is **optimal** if $f_0(x) = p^*$; $X_{\text{opt}}$ is the set of optimal points

$x$ is **locally optimal** if there is an $R > 0$ such that $x$ is optimal for

minimize (over $z$) $f_0(z)$

subject to $f_i(z) \leq 0$, $i = 1, \ldots, m$, $h_i(z) = 0$, $i = 1, \ldots, p$

$\|z - x\|_2 \leq R$

**examples** (with $n = 1$, $m = p = 0$)

- $f_0(x) = 1/x$, $\text{dom} f_0 = \mathbb{R}_{++}$: $p^* = 0$, no optimal point
- $f_0(x) = -\log x$, $\text{dom} f_0 = \mathbb{R}_{++}$: $p^* = -\infty$
- $f_0(x) = x \log x$, $\text{dom} f_0 = \mathbb{R}_{++}$: $p^* = -1/e$, $x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x$, $p^* = -\infty$, local optimum at $x = 1$
Implicit constraints

the standard form optimization problem has an implicit constraint

\[ x \in \mathcal{D} = \bigcap_{i=0}^{m} \text{dom} f_i \cap \bigcap_{i=1}^{p} \text{dom} h_i, \]

- we call \( \mathcal{D} \) the **domain** of the problem
- the constraints \( f_i(x) \leq 0, h_i(x) = 0 \) are the explicit constraints
- a problem is **unconstrained** if it has no explicit constraints \( (m = p = 0) \)

**example:**

minimize \[ f_0(x) = - \sum_{i=1}^{k} \log(b_i - a_i^T x) \]

is an unconstrained problem with implicit constraints \( a_i^T x < b_i \)
Feasibility problem

find \quad x
subject to \quad f_i(x) \leq 0, \quad i = 1, \ldots, m
h_i(x) = 0, \quad i = 1, \ldots, p

can be considered a special case of the general problem with \( f_0(x) = 0 \):

minimize \quad 0
subject to \quad f_i(x) \leq 0, \quad i = 1, \ldots, m
h_i(x) = 0, \quad i = 1, \ldots, p

\begin{itemize}
  \item \( p^* = 0 \) if constraints are feasible; any feasible \( x \) is optimal
  \item \( p^* = \infty \) if constraints are infeasible
\end{itemize}
Convex optimization problem

standard form convex optimization problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad a_i^T x = b_i, \quad i = 1, \ldots, p
\end{align*}
\]

- \( f_0, f_1, \ldots, f_m \) are convex; equality constraints are affine
- problem is \textit{quasiconvex} if \( f_0 \) is quasiconvex (and \( f_1, \ldots, f_m \) convex)

often written as

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

important property: feasible set of a convex optimization problem is convex
example

minimize \( f_0(x) = x_1^2 + x_2^2 \)
subject to \( f_1(x) = x_1/(1 + x_2^2) \leq 0 \)
\( h_1(x) = (x_1 + x_2)^2 = 0 \)

• \( f_0 \) is convex; feasible set \( \{(x_1, x_2) \mid x_1 = -x_2 \leq 0\} \) is convex

• not a convex problem (according to our definition): \( f_1 \) is not convex, \( h_1 \) is not affine

• equivalent (but not identical) to the convex problem

minimize \( x_1^2 + x_2^2 \)
subject to \( x_1 \leq 0 \)
\( x_1 + x_2 = 0 \)
Local and global optima

any locally optimal point of a convex problem is (globally) optimal

proof: suppose $x$ is locally optimal and $y$ is optimal with $f_0(y) < f_0(x)$

$x$ locally optimal means there is an $R > 0$ such that

$$z \text{ feasible, } \|z - x\|_2 \leq R \implies f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

• $\|y - x\|_2 > R$, so $0 < \theta < 1/2$

• $z$ is a convex combination of two feasible points, hence also feasible

• $\|z - x\|_2 = R/2$ and

$$f_0(z) \leq \theta f_0(x) + (1 - \theta)f_0(y) < f_0(x)$$

which contradicts our assumption that $x$ is locally optimal
Optimality criterion for differentiable $f_0$

$x$ is optimal if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0 \quad \text{for all feasible } y$$

if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set $X$ at $x$
\begin{itemize}
\item \textbf{unconstrained problem}: $x$ is optimal if and only if
\[
x \in \text{dom } f_0, \quad \nabla f_0(x) = 0
\]
\item \textbf{equality constrained problem}
\[
\text{minimize } f_0(x) \quad \text{subject to } \quad Ax = b
\]
x is optimal if and only if there exists a $\nu$ such that
\[
x \in \text{dom } f_0, \quad Ax = b, \quad \nabla f_0(x) + A^T \nu = 0
\]
\item \textbf{minimization over nonnegative orthant}
\[
\text{minimize } f_0(x) \quad \text{subject to } \quad x \succeq 0
\]
x is optimal if and only if
\[
x \in \text{dom } f_0, \quad x \succeq 0, \quad \begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}
\]
\end{itemize}
Equivalent convex problems

two problems are (informally) equivalent if the solution of one is readily obtained from the solution of the other, and vice-versa

some common transformations that preserve convexity:

• eliminating equality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize (over } z) & \quad f_0(Fz + x_0) \\
\text{subject to} & \quad f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

where \( F \) and \( x_0 \) are such that

\[
Ax = b \iff x = Fz + x_0 \text{ for some } z
\]
• introducing equality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(A_0x + b_0) \\
\text{subject to} & \quad f_i(A_ix + b_i) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize (over } x, y_i) & \quad f_0(y_0) \\
\text{subject to} & \quad f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\
y_i &= A_ix + b_i, \quad i = 0, 1, \ldots, m
\end{align*}
\]

• introducing slack variables for linear inequalities

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad a_i^Tx \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize (over } x, s) & \quad f_0(x) \\
\text{subject to} & \quad a_i^Tx + s_i = b_i, \quad i = 1, \ldots, m \\
s_i & \geq 0, \quad i = 1, \ldots, m
\end{align*}
\]
• **epigraph form**: standard form convex problem is equivalent to

\[
\begin{align*}
\text{minimize (over } x, t) & \quad t \\
\text{subject to} & \quad f_0(x) - t \leq 0 \\
& \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

• **minimizing over some variables**

\[
\begin{align*}
\text{minimize} & \quad f_0(x_1, x_2) \\
\text{subject to} & \quad f_i(x_1) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize} & \quad \tilde{f}_0(x_1) \\
\text{subject to} & \quad f_i(x_1) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

where \( \tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2) \)
Quasiconvex optimization

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, \quad i = 1, \ldots, m \)
\( Ax = b \)

with \( f_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) quasiconvex, \( f_1, \ldots, f_m \) convex

...can have locally optimal points that are not (globally) optimal...
convex representation of sublevel sets of $f_0$

if $f_0$ is quasiconvex, there exists a family of functions $\phi_t$ such that:

- $\phi_t(x)$ is convex in $x$ for fixed $t$
- $t$-sublevel set of $f_0$ is 0-sublevel set of $\phi_t$, i.e.,

$$f_0(x) \leq t \iff \phi_t(x) \leq 0$$

example

$$f_0(x) = \frac{p(x)}{q(x)}$$

with $p$ convex, $q$ concave, and $p(x) \geq 0$, $q(x) > 0$ on $\text{dom } f_0$

can take $\phi_t(x) = p(x) - tq(x)$:

- for $t \geq 0$, $\phi_t$ convex in $x$
- $p(x)/q(x) \leq t$ if and only if $\phi_t(x) \leq 0$
quasiconvex optimization via convex feasibility problems

\[ \phi_t(x) \leq 0, \quad f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad Ax = b \quad (1) \]

- for fixed \( t \), a convex feasibility problem in \( x \)
- if feasible, we can conclude that \( t \geq p^* \); if infeasible, \( t \leq p^* \)

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*Bisection method for quasiconvex optimization*

**given** \( l \leq p^*, u \geq p^* \), tolerance \( \epsilon > 0 \).

**repeat**

1. \( t := (l + u)/2 \).
2. Solve the convex feasibility problem (1).
3. **if** (1) is feasible, \( u := t \); **else** \( l := t \).

**until** \( u - l \leq \epsilon \).

---

requires exactly \( \lceil \log_2((u - l)/\epsilon) \rceil \) iterations (where \( u, l \) are initial values)
Linear program (LP)

\[
\begin{align*}
\text{minimize} & \quad c^T x + d \\
\text{subject to} & \quad Gx \leq h \\
& \quad Ax = b
\end{align*}
\]

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron
Examples

**diet problem:** choose quantities \( x_1, \ldots, x_n \) of \( n \) foods

- one unit of food \( j \) costs \( c_j \), contains amount \( a_{ij} \) of nutrient \( i \)
- healthy diet requires nutrient \( i \) in quantity at least \( b_i \)

to find cheapest healthy diet,

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b, \quad x \geq 0
\end{align*}
\]

**piecewise-linear minimization**

\[
\begin{align*}
\text{minimize} & \quad \max_{i=1, \ldots, m}(a_i^T x + b_i) \\
\end{align*}
\]

equivalent to an LP

\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad a_i^T x + b_i \leq t, \quad i = 1, \ldots, m
\end{align*}
\]
Chebyshev center of a polyhedron

Chebyshev center of

\[ \mathcal{P} = \{ x \mid a_i^T x \leq b_i, \ i = 1, \ldots, m \} \]

is center of largest inscribed ball

\[ \mathcal{B} = \{ x_c + u \mid \|u\|_2 \leq r \} \]

- \( a_i^T x \leq b_i \) for all \( x \in \mathcal{B} \) if and only if

\[
\sup \{ a_i^T (x_c + u) \mid \|u\|_2 \leq r \} = a_i^T x_c + r \|a_i\|_2 \leq b_i
\]

- hence, \( x_c, r \) can be determined by solving the LP

\[
\begin{align*}
\text{maximize} & \quad r \\
\text{subject to} & \quad a_i^T x_c + r \|a_i\|_2 \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]
(Generalized) linear-fractional program

\[
\begin{align*}
& \text{minimize} & & f_0(x) \\
& \text{subject to} & & Gx \leq h \\
& & & Ax = b
\end{align*}
\]

linear-fractional program

\[
f_0(x) = \frac{c^T x + d}{e^T x + f}, \quad \text{dom} \ f_0(x) = \{x \mid e^T x + f > 0\}
\]

- a quasiconvex optimization problem; can be solved by bisection
- also equivalent to the LP (variables \( y, z \))

\[
\begin{align*}
& \text{minimize} & & c^T y + dz \\
& \text{subject to} & & G y \leq h z \\
& & & A y = b z \\
& & & e^T y + f z = 1 \\
& & & z \geq 0
\end{align*}
\]
generalized linear-fractional program

\[ f_0(x) = \max_{i=1,\ldots,r} \frac{c_i^T x + d_i}{e_i^T x + f_i}, \quad \text{dom } f_0(x) = \{ x \mid e_i^T x + f_i > 0, \ i = 1, \ldots, r \} \]

a quasiconvex optimization problem; can be solved by bisection

**example:** Von Neumann model of a growing economy

maximize (over \( x, x^+ \)) \[ \min_{i=1,\ldots,n} \frac{x_i^+}{x_i} \]
subject to \( x^+ \geq 0, \quad Bx^+ \leq Ax \)

- \( x, x^+ \in \mathbb{R}^n \): activity levels of \( n \) sectors, in current and next period
- \( (Ax)_i, (Bx^+)_i \): produced, resp. consumed, amounts of good \( i \)
- \( x_i^+/x_i \): growth rate of sector \( i \)

allocate activity to maximize growth rate of slowest growing sector
Quadratic program (QP)

minimize \( (1/2)x^T P x + q^T x + r \)
subject to \( Gx \leq h \)
\( Ax = b \)

- \( P \in S^n_+ \), so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron
Examples

least-squares

\[
\text{minimize } \|Ax - b\|_2^2
\]

• analytical solution \(x^* = A^\dagger b\) (\(A^\dagger\) is pseudo-inverse)
• can add linear constraints, e.g., \(l \leq x \leq u\)

linear program with random cost

\[
\begin{align*}
\text{minimize} \quad & \bar{c}^T x + \gamma x^T \Sigma x = E \bar{c}^T x + \gamma \text{var}(c^T x) \\
\text{subject to} \quad & Gx \leq h, \quad Ax = b
\end{align*}
\]

• \(c\) is random vector with mean \(\bar{c}\) and covariance \(\Sigma\)
• hence, \(c^T x\) is random variable with mean \(\bar{c}^T x\) and variance \(x^T \Sigma x\)
• \(\gamma > 0\) is risk aversion parameter; controls the trade-off between expected cost and variance (risk)
Quadratically constrained quadratic program (QCQP)

minimize \((1/2)x^TP_0x + q_0^Tx + r_0\)
subject to \((1/2)x^TP_ix + q_i^Tx + r_i \leq 0, \quad i = 1, \ldots, m\)
\[Ax = b\]

- \(P_i \in S^n_+\); objective and constraints are convex quadratic

- if \(P_1, \ldots, P_m \in S^n_{++}\), feasible region is intersection of \(m\) ellipsoids and an affine set
Second-order cone programming

minimize \( f^T x \)
subject to \( \| A_i x + b_i \|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \)
\( Fx = g \)

\((A_i \in \mathbb{R}^{n_i \times n}, F \in \mathbb{R}^{p \times n})\)

- inequalities are called second-order cone (SOC) constraints:

\((A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbb{R}^{n_i+1}\)

- for \( n_i = 0 \), reduces to an LP; if \( c_i = 0 \), reduces to a QCQP

- more general than QCQP and LP
Robust linear programming

the parameters in optimization problems are often uncertain, e.g., in an LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, m,
\end{align*}
\]

there can be uncertainty in \(c, a_i, b_i\)

two common approaches to handling uncertainty (in \(a_i\), for simplicity)

- deterministic model: constraints must hold for all \(a_i \in \mathcal{E}_i\)

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m,
\end{align*}
\]

- stochastic model: \(a_i\) is random variable; constraints must hold with probability \(\eta\)

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \text{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m
\end{align*}
\]
deterministic approach via SOCP

- choose an ellipsoid as $\mathcal{E}_i$:

$$\mathcal{E}_i = \{ \bar{a}_i + P_i u \mid \|u\|_2 \leq 1 \} \quad (\bar{a}_i \in \mathbb{R}^n, \quad P_i \in \mathbb{R}^{n \times n})$$

center is $\bar{a}_i$, semi-axes determined by singular values/vectors of $P_i$

- robust LP

minimize $c^T x$
subject to $a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m$

is equivalent to the SOCP

minimize $c^T x$
subject to $\bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m$

(follows from $\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$)
stochastic approach via SOCP

- assume $a_i$ is Gaussian with mean $\bar{a}_i$, covariance $\Sigma_i$ ($a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$)
- $a_i^T x$ is Gaussian r.v. with mean $\bar{a}_i^T x$, variance $x^T \Sigma_i x$; hence

$$\text{prob}(a_i^T x \leq b_i) = \Phi \left( \frac{b_i - \bar{a}_i^T x}{\| \Sigma_i^{1/2} x \|_2} \right)$$

where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2} dt$ is CDF of $\mathcal{N}(0, 1)$

- robust LP

$$\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & \text{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m,
\end{array}$$

with $\eta \geq 1/2$, is equivalent to the SOCP

$$\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \| \Sigma_i^{1/2} x \|_2 \leq b_i, \quad i = 1, \ldots, m
\end{array}$$
Geometric programming

monomial function

\[ f(x) = cx_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}, \quad \text{dom } f = \mathbb{R}^n_+ \]

with \( c > 0; \) exponent \( \alpha_i \) can be any real number

posynomial function: sum of monomials

\[ f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}}x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \quad \text{dom } f = \mathbb{R}^n_+ \]

geometric program (GP)

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 1, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 1, \quad i = 1, \ldots, p
\end{align*}
\]

with \( f_i \) posynomial, \( h_i \) monomial
**Geometric program in convex form**

change variables to $y_i = \log x_i$, and take logarithm of cost, constraints

- monomial $f(x) = cx_1^{a_1} \cdots x_n^{a_n}$ transforms to
  \[
  \log f(e^{y_1}, \ldots, e^{y_n}) = a^T y + b \quad (b = \log c)
  \]

- posynomial $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to
  \[
  \log f(e^{y_1}, \ldots, e^{y_n}) = \log \left( \sum_{k=1}^{K} e^{a_k^T y + b_k} \right) \quad (b_k = \log c_k)
  \]

- geometric program transforms to convex problem

  minimize \quad \log \left( \sum_{k=1}^{K} \exp(a_{0k}^T y + b_{0k}) \right)
  
  subject to \quad \log \left( \sum_{k=1}^{K} \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \ldots, m
  
  $Gy + d = 0$
Design of cantilever beam

- $N$ segments with unit lengths, rectangular cross-sections of size $w_i \times h_i$
- given vertical force $F$ applied at the right end

**design problem**

minimize total weight

subject to upper & lower bounds on $w_i, h_i$

upper bound & lower bounds on aspect ratios $h_i/w_i$

upper bound on stress in each segment

upper bound on vertical deflection at the end of the beam

variables: $w_i, h_i$ for $i = 1, \ldots, N$
objective and constraint functions

- total weight $w_1 h_1 + \cdots + w_N h_N$ is posynomial

- aspect ratio $h_i / w_i$ and inverse aspect ratio $w_i / h_i$ are monomials

- maximum stress in segment $i$ is given by $6i F / (w_i h_i^2)$, a monomial

- the vertical deflection $y_i$ and slope $v_i$ of central axis at the right end of segment $i$ are defined recursively as

$$
v_i = 12(i - 1/2) \frac{F}{E w_i h_i^3} + v_{i+1}
$$

$$
y_i = 6(i - 1/3) \frac{F}{E w_i h_i^3} + v_{i+1} + y_{i+1}
$$

for $i = N, N - 1, \ldots, 1$, with $v_{N+1} = y_{N+1} = 0$ ($E$ is Young’s modulus)

$v_i$ and $y_i$ are posynomial functions of $w$, $h$
formulation as a GP

\[
\begin{align*}
\text{minimize} & \quad w_1 h_1 + \cdots + w_N h_N \\
\text{subject to} & \quad w^{-1}_{\text{max}} w_i \leq 1, \quad w_{\text{min}} w^{-1}_i \leq 1, \quad i = 1, \ldots, N \\
& \quad h^{-1}_{\text{max}} h_i \leq 1, \quad h_{\text{min}} h^{-1}_i \leq 1, \quad i = 1, \ldots, N \\
& \quad S^{-1}_{\text{max}} w^{-1}_i h_i \leq 1, \quad S_{\text{min}} w_i h^{-1}_i \leq 1, \quad i = 1, \ldots, N \\
& \quad 6i F \sigma^{-1}_{\text{max}} w^{-1}_i h^{-2}_i \leq 1, \quad i = 1, \ldots, N \\
& \quad y^{-1}_{\text{max}} y_1 \leq 1
\end{align*}
\]

note

- we write \( w_{\text{min}} \leq w_i \leq w_{\text{max}} \) and \( h_{\text{min}} \leq h_i \leq h_{\text{max}} \)

\[
\begin{align*}
\frac{w_{\text{min}}}{w_i} \leq 1, \quad \frac{w_i}{w_{\text{max}}} \leq 1, \quad \frac{h_{\text{min}}}{h_i} \leq 1, \quad \frac{h_i}{h_{\text{max}}} \leq 1
\end{align*}
\]

- we write \( S_{\text{min}} \leq h_i/w_i \leq S_{\text{max}} \) as

\[
\begin{align*}
S_{\text{min}} w_i/h_i \leq 1, \quad h_i/(w_i S_{\text{max}}) \leq 1
\end{align*}
\]
Minimizing spectral radius of nonnegative matrix

Perron-Frobenius eigenvalue \( \lambda_{\text{pf}}(A) \)

- exists for (elementwise) positive \( A \in \mathbb{R}^{n \times n} \)
- a real, positive eigenvalue of \( A \), equal to spectral radius \( \max_i |\lambda_i(A)| \)
- determines asymptotic growth (decay) rate of \( A^k: A^k \sim \lambda_{\text{pf}}^k \) as \( k \to \infty \)
- alternative characterization: \( \lambda_{\text{pf}}(A) = \inf \{ \lambda \mid Av \preceq \lambda v \text{ for some } v \succ 0 \} \)

minimizing spectral radius of matrix of posynomials

- minimize \( \lambda_{\text{pf}}(A(x)) \), where the elements \( A(x)_{ij} \) are posynomials of \( x \)
- equivalent geometric program:

\[
\begin{align*}
\text{minimize} & \quad \lambda \\
\text{subject to} & \quad \sum_{j=1}^{n} A(x)_{ij} v_j / (\lambda v_i) \leq 1, \quad i = 1, \ldots, n
\end{align*}
\]

variables \( \lambda, v, x \)
Generalized inequality constraints

convex problem with generalized inequality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \preceq_K 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

- \( f_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) convex; \( f_i : \mathbb{R}^n \rightarrow \mathbb{R}^{k_i} \) \( K_i \)-convex w.r.t. proper cone \( K_i \)
- same properties as standard convex problem (convex feasible set, local optimum is global, etc.)

conic form problem: special case with affine objective and constraints

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Fx + g \preceq_K 0 \\
& \quad Ax = b
\end{align*}
\]

extends linear programming \((K = \mathbb{R}_+^m)\) to nonpolyhedral cones
Semidefinite program (SDP)

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\
& \quad Ax = b
\end{align*}
\]

with \( F_i, G \in S^k \)

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with multiple LMI constraints: for example,

\[
x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \quad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0
\]

is equivalent to single LMI

\[
x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \hat{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \hat{F}_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \hat{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0
\]
LP and SOCP as SDP

LP and equivalent SDP

LP: \[ \text{minimize} \quad c^T x \]
\[ \text{subject to} \quad Ax \preceq b \]
SDP: \[ \text{minimize} \quad c^T x \]
\[ \text{subject to} \quad \text{diag}(Ax - b) \preceq 0 \]

(note different interpretation of generalized inequality $\preceq$)

SOCP and equivalent SDP

SOCP: \[ \text{minimize} \quad f^T x \]
\[ \text{subject to} \quad \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \]

SDP: \[ \text{minimize} \quad f^T x \]
\[ \text{subject to} \quad \begin{bmatrix} (c_i^T x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, m \]
Eigenvalue minimization

minimize $\lambda_{\text{max}}(A(x))$

where $A(x) = A_0 + x_1A_1 + \cdots + x_nA_n$ (with given $A_i \in \mathbb{S}^k$)

equivalent SDP

minimize $t$
subject to $A(x) \preceq tI$

• variables $x \in \mathbb{R}^n$, $t \in \mathbb{R}$
• follows from

$\lambda_{\text{max}}(A) \leq t \iff A \preceq tI$
Matrix norm minimization

\[
\text{minimize } \|A(x)\|_2 = \left(\lambda_{\text{max}}(A(x)^T A(x))\right)^{1/2}
\]

where \( A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \) (with given \( A_i \in \mathbb{R}^{p \times q} \))

equivalent SDP

\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0
\end{align*}
\]

- variables \( x \in \mathbb{R}^n, \ t \in \mathbb{R} \)
- constraint follows from

\[
\|A\|_2 \leq t \iff A^T A \leq t^2 I, \quad t \geq 0
\]

\[
\iff \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succeq 0
\]
Vector optimization

general vector optimization problem

\[
\begin{align*}
\text{minimize (w.r.t. } K) & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) \leq 0, \quad i = 1, \ldots, p
\end{align*}
\]

vector objective \( f_0 : \mathbb{R}^n \to \mathbb{R}^q \), minimized w.r.t. proper cone \( K \in \mathbb{R}^q \)

convex vector optimization problem

\[
\begin{align*}
\text{minimize (w.r.t. } K) & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

with \( f_0 \ K\)-convex, \( f_1, \ldots, f_m \) convex
Optimal and Pareto optimal points

set of achievable objective values

\[ \mathcal{O} = \{ f_0(x) \mid x \text{ feasible} \} \]

- feasible \( x \) is **optimal** if \( f_0(x) \) is a minimum value of \( \mathcal{O} \)
- feasible \( x \) is **Pareto optimal** if \( f_0(x) \) is a minimal value of \( \mathcal{O} \)
Multicriterion optimization

vector optimization problem with $K = \mathbb{R}_+^q$

$$f_0(x) = (F_1(x), \ldots, F_q(x))$$

• $q$ different objectives $F_i$; roughly speaking we want all $F_i$’s to be small

• feasible $x^*$ is optimal if

$$y \text{ feasible} \implies f_0(x^*) \preceq f_0(y)$$

if there exists an optimal point, the objectives are noncompeting

• feasible $x^{po}$ is Pareto optimal if

$$y \text{ feasible, } f_0(y) \preceq f_0(x^{po}) \implies f_0(x^{po}) = f_0(y)$$

if there are multiple Pareto optimal values, there is a trade-off between the objectives
Regularized least-squares

minimize (w.r.t. $\mathbb{R}^2_+$) \((\|Ax - b\|^2_2, \|x\|^2_2)\)

example for $A \in \mathbb{R}^{100 \times 10}$; heavy line is formed by Pareto optimal points
Risk return trade-off in portfolio optimization

minimize (w.r.t. $\mathbf{R}_+^2$) $(\mathbf{p^T x}, x^T \Sigma x)$
subject to $\mathbf{1}^T x = 1, \quad x \geq 0$

- $x \in \mathbf{R}^n$ is investment portfolio; $x_i$ is fraction invested in asset $i$
- $\mathbf{p} \in \mathbf{R}^n$ is vector of relative asset price changes; modeled as a random variable with mean $\mathbf{p}$, covariance $\Sigma$
- $\mathbf{p^T x} = \mathbf{E} r$ is expected return; $x^T \Sigma x = \mathbf{var} r$ is return variance

example

Convex optimization problems
Scalarization

to find Pareto optimal points: choose $\lambda \succ K^* 0$ and solve scalar problem

\[
\text{minimize} \quad \lambda^T f_0(x) \\
\text{subject to} \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
h_i(x) = 0, \quad i = 1, \ldots, p
\]

if $x$ is optimal for scalar problem, then it is Pareto-optimal for vector optimization problem

for convex vector optimization problems, can find (almost) all Pareto optimal points by varying $\lambda \succ K^* 0$
Scalarization for multicriterion problems

to find Pareto optimal points, minimize positive weighted sum

$$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_q F_q(x)$$

examples

• regularized least-squares problem of page 4–43

  take \( \lambda = (1, \gamma) \) with \( \gamma > 0 \)

  minimize \( \|Ax - b\|_2^2 + \gamma\|x\|_2^2 \)

  for fixed \( \gamma \), a LS problem
• risk-return trade-off of page 4–44

minimize \(-\bar{p}^T x + \gamma x^T \Sigma x\)
subject to \(1^T x = 1, \quad x \succeq 0\)

for fixed \(\gamma > 0\), a quadratic program