Optimization I

Final **MATH 6366-15624 (Fall 2013)** **December 5-8, 2013**

Name: ______________________________

- This is a 72 hour take-home final. Please turn it in by electronic means via email to jiwenhe@math.uh.edu by 3PM on Sunday, December 8, i.e., 72 hours after you receive it.
- When sending your work by email, please **type out and follow by your name** the honors pledge "**I have neither given nor received assistance on this exam.**"
- The final is not supposed to take anything close to 72 hours; I am giving you 72 hours so you can take a number of long breaks (to eat, sleep, study for other classes, takes other final exams, etc.).
- You may use any books, notes, but you may not discuss the exam with anyone until December 8, after everyone has taken the exam.
- Since this is a take home exam, please respect the honor code.
- The problems are all equally weighted. This is to make things simple, and not because the problems are all of equal difficulty.
- Keep in mind that there are multiple approaches (and valid solutions) to some of the problems, and these approaches can differ considerably in complexity. Also, there are several tricky parts and at least one tricky problem, so don't expect to be able to solve everything.
- You will be graded on clarity and conciseness as well as accuracy and correctness. Please take the time and make the effort to make your solutions clear. Please try to use standard and simple notation, introducing only as many new symbols as is absolutely necessary. The solutions are not long, so if you find that your solution to a problem goes on and on for pages, you should try to figure out a simpler one. I expect neat, legible exams from everyone.

20 points

1. Show that if S_1 and S_2 are convex sets in $R^{m\times n}$, then so is their partial sum

$$S = \{\, (x, y_1 + y_2) \,|\, x \in R^m, y_1 \in R^n, y_2 \in R^n, (x, y_1) \in S_1, (x, y_2) \in S_2 \,\}.$$

20 points

2. Show that a continuous function $f : R^n \to R$ is convex if and only if for every line segment, its average value on the segment is less than or equal to the average of its values at the endpoints of the segment: For every $x, y \in R^n$,

$$\int_0^1 f(x + \lambda(y - x))d\lambda \le \frac{f(x) + f(y)}{2}.$$

20 points

3. Suppose $p < 1$, $p \neq 0$. Show that the function

$$f(x) = \left(\sum_{i=1}^n x_i^p\right)^{1/p}$$

with $\mathrm{dom} f = R^n_{++}$ is concave. This includes as special cases $f(x) = \left(\sum_{i=1}^n x_i^{1/2}\right)^2$ and the harmonic mean $f(x) = \left(\sum_{i=1}^n 1/x_i\right)^{-1}$.

20 points

4. Consider a network of n nodes, with directed links connecting each pair of nodes. The variables in the problem are the flows on each link: x_{ij} will denote the flow from node i to node j. The cost of the flow along the link from node i to node j is given by $c_{ij}x_{ij}$, where c_{ij} are given constants. The total cost across the network is

$$C = \sum_{i,j=1}^n c_{ij}x_{ij}.$$

Each link flow x_{ij} is also subject to a given lower bound l_{ij} (usually assumed to be nonnegative) and an upper bound u_{ij}.

The external supply at node i is given by b_i, where $b_i > 0$ means an external flow enters the network at node i, and $b_i < 0$ means that at node i, an amount $|b_i|$ flows out of the network. We assume that $1^T b = 0$, i.e., the total external supply equals total external demand. At each node we have conservation of flow: the total flow into node i along links and the external supply, minus the total flow out along the links, equals zero.

The problem is to minimize the total cost of flow through the network, subject to the constraints described above. Formulate this problem as an LP.

20 points

5. In a Boolean linear program, the variable x is constrained to have components equal to zero or one:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \\ & x_i \in \{0,1\}, \quad i = 1,\ldots,n. \end{array} \qquad (1)$$

In general, such problems are very difficult to solve, even though the feasible set is finite (containing at most 2^n points).

In a general method called relaxation, the constraint that x_i be zero or one is replaced with the linear inequalities $0 \le x_i \le 1$:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \\ & 0 \le x_i \le 1, \quad i = 1, \ldots, n. \end{array} \tag{2}$$

We refer to this problem as the LP relaxation of the Boolean LP (1). The LP relaxation is far easier to solve than the original Boolean LP.

(a) Show that the optimal value of the LP relaxation (2) is a lower bound on the optimal value of the Boolean LP (1). What can you say about the Boolean LP if the LP relaxation is infeasible?

(b) It sometimes happens that the LP relaxation has a solution with $x_i \in \{0, 1\}$. What can you say in this case?

(c) Derive the Lagrange dual of the LP relaxation (2).

20 points

6. We consider the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array} \tag{3}$$

where $f : R^n \to R$ is convex and differentiable, and $A \in R^{m \times n}$ with rank$A = m$.

In a quadratic penalty method, we form an auxiliary function

$$\phi(x) = f(x) + \alpha \|Ax - b\|_2^2$$

where $\alpha > 0$ is a parameter. This auxiliary function consists of the objective plus the penalty term $\alpha \|Ax - b\|_2^2$. The idea is that a minimizer of the auxiliary function, $\tilde{x}$, should be an approximate solution of the original problem. Intuition suggests that the larger the penalty weight α, the better the approximation $\tilde{x}$ to a solution of the original problem.

Suppose $\tilde{x}$ is a minimizer of ϕ. Show how to find, from $\tilde{x}$, a dual feasible point for (3). Find the corresponding lower bound on the optimal value of (3).

20 points

7. Consider the QCQP

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & (x_1 - 1)^2 + (x_2 - 1)^2 \le 1 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \le 1 \end{array} \tag{4}$$

with variable $x \in R^2$.

(a) Sketch the feasible set and level sets of the objective. Find the optimal point x^* and optimal value p^*.

(b) Give the KKT conditions. Do there exist Lagrange multipliers λ_1^* and λ_2^* that prove that x^* is optimal?

(c) Derive and solve the Lagrange dual problem. Does strong duality hold?

20 points 8. We consider the equality constrained problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array} \tag{5}$$

where $f : R^n \to R$ is convex and twice differentiable, and $A \in R^{m\times n}$ with $\text{rank}A = m$.

(a) Derive the Newton step Δx_{nt} for the problem (5).

(b) Suppose $Q \geq 0$. The problem

$$\begin{array}{ll} \text{minimize} & f(x) + (Ax-b)^T Q(Ax-b) \\ \text{subject to} & Ax = b \end{array} \tag{6}$$

is equivalent to the original equality constrained optimization problem (5). Is the Newton step for this problem the same as the Newton step for the original problem?