12. Interior-point methods

- inequality constrained minimization
- logarithmic barrier function and central path
- barrier method
- feasibility and phase I methods
- complexity analysis via self-concordance
- generalized inequalities
Inequality constrained minimization

minimize \quad f_0(x) \\
subject to \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
Ax = b \tag{1}

- \( f_i \) convex, twice continuously differentiable
- \( A \in \mathbb{R}^{p \times n} \) with \( \text{rank} \ A = p \)
- we assume \( p^* \) is finite and attained
- we assume problem is strictly feasible: there exists \( \tilde{x} \) with

\[
\tilde{x} \in \text{dom} \ f_0, \quad f_i(\tilde{x}) < 0, \quad i = 1, \ldots, m, \quad A\tilde{x} = b
\]

hence, strong duality holds and dual optimum is attained
Examples

• LP, QP, QCQP, GP

• entropy maximization with linear inequality constraints

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{n} x_i \log x_i \\
\text{subject to} & \quad Fx \leq g \\
& \quad Ax = b
\end{align*}
\]

with \( \text{dom} \, f_0 = \mathbb{R}^{n}_{++} \)

• differentiability may require reformulating the problem, \( e.g. \), piecewise-linear minimization or \( \ell_{\infty} \)-norm approximation via LP

• SDPs and SOCPs are better handled as problems with generalized inequalities (see later)
Logarithmic barrier

reformulation of (1) via indicator function:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) + \sum_{i=1}^{m} I_-(f_i(x)) \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

where \( I_-(u) = 0 \) if \( u \leq 0 \), \( I_-(u) = \infty \) otherwise (indicator function of \( \mathbb{R}_- \))

approximation via logarithmic barrier

\[
\begin{align*}
\text{minimize} & \quad f_0(x) - \frac{1}{t} \sum_{i=1}^{m} \log(-f_i(x)) \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

- an equality constrained problem
- for \( t > 0 \), \( -\frac{1}{t} \log(-u) \) is a smooth approximation of \( I_- \)
- approximation improves as \( t \to \infty \)
logarithmic barrier function

\[ \phi(x) = - \sum_{i=1}^{m} \log(-f_i(x)), \quad \text{dom} \phi = \{ x \mid f_1(x) < 0, \ldots, f_m(x) < 0 \} \]

- convex (follows from composition rules)
- twice continuously differentiable, with derivatives

\[ \nabla \phi(x) = \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) \]

\[ \nabla^2 \phi(x) = \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x) \]
Central path

• for \( t > 0 \), define \( x^*(t) \) as the solution of

\[
\begin{align*}
\text{minimize} & \quad tf_0(x) + \phi(x) \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

(for now, assume \( x^*(t) \) exists and is unique for each \( t > 0 \))

• central path is \( \{x^*(t) \mid t > 0\} \)

**example:** central path for an LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, 6
\end{align*}
\]

hyperplane \( c^T x = c^T x^*(t) \) is tangent to level curve of \( \phi \) through \( x^*(t) \)
Dual points on central path

\[ x = x^*(t) \] if there exists a \( w \) such that

\[
t \nabla f_0(x) + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) + ATw = 0, \quad Ax = b
\]

• therefore, \( x^*(t) \) minimizes the Lagrangian

\[
L(x, \lambda^*(t), \nu^*(t)) = f_0(x) + \sum_{i=1}^{m} \lambda_i^*(t) f_i(x) + \nu^*(t)^T (Ax - b)
\]

where we define \( \lambda_i^*(t) = 1/(-tf_i(x^*(t))) \) and \( \nu^*(t) = w/t \)

• this confirms the intuitive idea that \( f_0(x^*(t)) \to p^* \) if \( t \to \infty \):

\[
p^* \geq g(\lambda^*(t), \nu^*(t)) = L(x^*(t), \lambda^*(t), \nu^*(t)) = f_0(x^*(t)) - m/t
\]
Interpretation via KKT conditions

\[ x = x^*(t), \lambda = \lambda^*(t), \nu = \nu^*(t) \text{ satisfy} \]

1. primal constraints: \( f_i(x) \leq 0, i = 1, \ldots, m, Ax = b \)
2. dual constraints: \( \lambda \geq 0 \)
3. approximate complementary slackness: \(-\lambda_i f_i(x) = 1/t, i = 1, \ldots, m\)
4. gradient of Lagrangian with respect to \( x \) vanishes:

\[
\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + A^T \nu = 0
\]

difference with KKT is that condition 3 replaces \( \lambda_i f_i(x) = 0 \)
Force field interpretation

**centering problem** (for problem with no equality constraints)

minimize \( tf_0(x) - \sum_{i=1}^{m} \log(-f_i(x)) \)

**force field interpretation**

- \( tf_0(x) \) is potential of force field \( F_0(x) = -t\nabla f_0(x) \)
- \( -\log(-f_i(x)) \) is potential of force field \( F_i(x) = (1/f_i(x))\nabla f_i(x) \)

the forces balance at \( x^*(t) \):

\[
F_0(x^*(t)) + \sum_{i=1}^{m} F_i(x^*(t)) = 0
\]
example

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

- objective force field is constant: \( F_0(x) = -tc \)
- constraint force field decays as inverse distance to constraint hyperplane:

\[
F_i(x) = \frac{-a_i}{b_i - a_i^T x}, \quad \|F_i(x)\|_2 = \frac{1}{\text{dist}(x, \mathcal{H}_i)}
\]

where \( \mathcal{H}_i = \{x \mid a_i^T x = b_i\} \)
**Barrier method**

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given strictly feasible $x$, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$.

repeat

1. *Centering step.* Compute $x^*(t)$ by minimizing $tf_0 + \phi$, subject to $Ax = b$.
2. *Update.* $x := x^*(t)$.
3. *Stopping criterion.* **quit** if $m/t < \epsilon$.
4. *Increase t.* $t := \mu t$.

---

- terminates with $f_0(x) - p^* \leq \epsilon$ (stopping criterion follows from $f_0(x^*(t)) - p^* \leq m/t$)
- centering usually done using Newton’s method, starting at current $x$
- choice of $\mu$ involves a trade-off: large $\mu$ means fewer outer iterations, more inner (Newton) iterations; typical values: $\mu = 10–20$
- several heuristics for choice of $t^{(0)}$
Convergence analysis

number of outer (centering) iterations: exactly

\[
\left\lceil \frac{\log(m/(\epsilon t(0)))}{\log \mu} \right\rceil
\]

plus the initial centering step (to compute \( x^*(t(0)) \))

centering problem

\[
\text{minimize } \; tf_0(x) + \phi(x)
\]

see convergence analysis of Newton’s method

- \( tf_0 + \phi \) must have closed sublevel sets for \( t \geq t(0) \)
- classical analysis requires strong convexity, Lipschitz condition
- analysis via self-concordance requires self-concordance of \( tf_0 + \phi \)
Examples

Inequality form LP \((m = 100\) inequalities, \(n = 50\) variables\)

- starts with \(x\) on central path \((t^{(0)} = 1\), duality gap 100\)
- terminates when \(t = 10^8\) \((\text{gap } 10^{-6})\)
- centering uses Newton’s method with backtracking
- total number of Newton iterations not very sensitive for \(\mu \geq 10\)
**geometric program** ($m = 100$ inequalities and $n = 50$ variables)

minimize \[ \log \left( \sum_{k=1}^{5} \exp(a_{0k}^T x + b_{0k}) \right) \]

subject to \[ \log \left( \sum_{k=1}^{5} \exp(a_{ik}^T x + b_{ik}) \right) \leq 0, \quad i = 1, \ldots, m \]
family of standard LPs ($A \in \mathbb{R}^{m \times 2m}$)

minimize $c^T x$
subject to $Ax = b$, $x \geq 0$

$m = 10, \ldots, 1000$; for each $m$, solve 100 randomly generated instances

number of iterations grows very slowly as $m$ ranges over a $100 : 1$ ratio
Feasibility and phase I methods

feasibility problem: find \( x \) such that

\[
f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad Ax = b \tag{2}
\]

phase I: computes strictly feasible starting point for barrier method

basic phase I method

\[
\begin{align*}
\text{minimize (over } x, s) \quad & s \\
\text{subject to} \quad & f_i(x) \leq s, \quad i = 1, \ldots, m \\
& Ax = b
\end{align*} \tag{3}
\]

- if \( x, s \) feasible, with \( s < 0 \), then \( x \) is strictly feasible for (2)
- if optimal value \( \bar{p}^* \) of (3) is positive, then problem (2) is infeasible
- if \( \bar{p}^* = 0 \) and attained, then problem (2) is feasible (but not strictly);
  - if \( \bar{p}^* = 0 \) and not attained, then problem (2) is infeasible
sum of infeasibilities phase I method

\[
\begin{align*}
\text{minimize} & \quad 1^T s \\
\text{subject to} & \quad s \geq 0, \quad f_i(x) \leq s_i, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

for infeasible problems, produces a solution that satisfies many more inequalities than basic phase I method

example (infeasible set of 100 linear inequalities in 50 variables)

left: basic phase I solution; satisfies 39 inequalities
right: sum of infeasibilities phase I solution; satisfies 79 solutions
**example:** family of linear inequalities $Ax \leq b + \gamma \Delta b$

- data chosen to be strictly feasible for $\gamma > 0$, infeasible for $\gamma \leq 0$
- use basic phase I, terminate when $s < 0$ or dual objective is positive

![Graph showing iterations vs. gamma](image)

Number of iterations roughly proportional to $\log(1/|\gamma|)$
Complexity analysis via self-concordance

same assumptions as on page 12–2, plus:

• sublevel sets (of $f_0$, on the feasible set) are bounded
• $tf_0 + \phi$ is self-concordant with closed sublevel sets

second condition

• holds for LP, QP, QCQP
• may require reformulating the problem, e.g.,

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{n} x_i \log x_i \\
\text{subject to} & \quad Fx \leq g
\end{align*}
\]

\[
\begin{align*}
\rightarrow \quad \text{minimize} & \quad \sum_{i=1}^{n} x_i \log x_i \\
\text{subject to} & \quad Fx \leq g, \quad x \geq 0
\end{align*}
\]

• needed for complexity analysis; barrier method works even when self-concordance assumption does not apply
**Newton iterations per centering step:** from self-concordance theory

\[
\#\text{Newton iterations} \leq \frac{\mu tf_0(x) + \phi(x) - \mu tf_0(x^+) - \phi(x^+)}{\gamma} + c
\]

- bound on effort of computing \( x^+ = x^*(\mu t) \) starting at \( x = x^*(t) \)
- \( \gamma, c \) are constants (depend only on Newton algorithm parameters)
- from duality (with \( \lambda = \lambda^*(t), \nu = \nu^*(t) \)):

\[
\mu tf_0(x) + \phi(x) - \mu tf_0(x^+) - \phi(x^+)
\]

\[
= \mu tf_0(x) - \mu tf_0(x^+) + \sum_{i=1}^{m} \log(-\mu t\lambda_i f_i(x^+)) - m \log \mu
\]

\[
\leq \mu tf_0(x) - \mu tf_0(x^+) - \mu t \sum_{i=1}^{m} \lambda_i f_i(x^+) - m - m \log \mu
\]

\[
\leq \mu tf_0(x) - \mu tg(\lambda, \nu) - m - m \log \mu
\]

\[
= m(\mu - 1 - \log \mu)
\]
total number of Newton iterations (excluding first centering step)

\[
\#\text{Newton iterations} \leq N = \left\lceil \frac{\log(m/(t(0)\varepsilon))}{\log \mu} \right\rceil \left( \frac{m(\mu - 1 - \log \mu)}{\gamma} + c \right)
\]

The figure shows \( N \) for typical values of \( \gamma, c \),

\[
m = 100, \quad \frac{m}{t(0)\varepsilon} = 10^5
\]

- confirms trade-off in choice of \( \mu \)
- in practice, \#iterations is in the tens; not very sensitive for \( \mu \geq 10 \)
polynomial-time complexity of barrier method

- for $\mu = 1 + 1/\sqrt{m}$:

$$N = O \left( \sqrt{m} \log \left( \frac{m/t^{(0)}}{\epsilon} \right) \right)$$

- number of Newton iterations for fixed gap reduction is $O(\sqrt{m})$

- multiply with cost of one Newton iteration (a polynomial function of problem dimensions), to get bound on number of flops

this choice of $\mu$ optimizes worst-case complexity; in practice we choose $\mu$ fixed ($\mu = 10, \ldots, 20$)
Generalized inequalities

minimize \quad f_0(x) \\
subject to \quad f_i(x) \preceq K_i 0, \quad i = 1, \ldots, m \\
\quad Ax = b

- \( f_0 \) convex, \( f_i : \mathbb{R}^n \rightarrow \mathbb{R}^{k_i}, \ i = 1, \ldots, m \), convex with respect to proper cones \( K_i \subseteq \mathbb{R}^{k_i} \)
- \( f_i \) twice continuously differentiable
- \( A \in \mathbb{R}^{p \times n} \) with rank \( A = p \)
- we assume \( p^* \) is finite and attained
- we assume problem is strictly feasible; hence strong duality holds and dual optimum is attained

examples of greatest interest: SOCP, SDP
Generalized logarithm for proper cone

$\psi : \mathbb{R}^q \rightarrow \mathbb{R}$ is generalized logarithm for proper cone $K \subseteq \mathbb{R}^q$ if:

- $\text{dom} \psi = \text{int} K$ and $\nabla^2 \psi(y) < 0$ for $y \succ_K 0$
- $\psi(sy) = \psi(y) + \theta \log s$ for $y \succ_K 0$, $s > 0$ ($\theta$ is the degree of $\psi$)

**examples**

- nonnegative orthant $K = \mathbb{R}^n_+$: $\psi(y) = \sum_{i=1}^n \log y_i$, with degree $\theta = n$
- positive semidefinite cone $K = S^n_+$:

  $\psi(Y) = \log \det Y \quad (\theta = n)$

- second-order cone $K = \{y \in \mathbb{R}^{n+1} \mid (y_1^2 + \cdots + y_n^2)^{1/2} \leq y_{n+1}\}$:

  $\psi(y) = \log(y_{n+1}^2 - y_1^2 - \cdots - y_n^2) \quad (\theta = 2)$

Interior-point methods
**properties** (without proof): for \( y \succsim_K 0, \)

\[
\nabla \psi(y) \succeq_{K^*} 0, \quad y^T \nabla \psi(y) = \theta
\]

- nonnegative orthant \( \mathbb{R}^n_+ \): \( \psi(y) = \sum_{i=1}^n \log y_i \)

\[
\nabla \psi(y) = (1/y_1, \ldots, 1/y_n), \quad y^T \nabla \psi(y) = n
\]

- positive semidefinite cone \( \mathbb{S}^n_+ \): \( \psi(Y) = \log \det Y \)

\[
\nabla \psi(Y) = Y^{-1}, \quad \text{tr}(Y \nabla \psi(Y)) = n
\]

- second-order cone \( K = \{ y \in \mathbb{R}^{n+1} \mid (y_1^2 + \cdots + y_n^2)^{1/2} \leq y_{n+1} \} \):

\[
\nabla \psi(y) = \frac{2}{y_{n+1}^2 - y_1^2 - \cdots - y_n^2} \begin{bmatrix} -y_1 \\ \vdots \\ -y_n \\ y_{n+1} \end{bmatrix}, \quad y^T \nabla \psi(y) = 2
\]
Logarithmic barrier and central path

**logarithmic barrier** for $f_1(x) \leq_{K_1} 0, \ldots, f_m(x) \leq_{K_m} 0$:

$$
\phi(x) = - \sum_{i=1}^{m} \psi_i(-f_i(x)), \quad \text{dom} \phi = \{ x \mid f_i(x) \prec_{K_i} 0, \ i = 1, \ldots, m \}
$$

- $\psi_i$ is generalized logarithm for $K_i$, with degree $\theta_i$
- $\phi$ is convex, twice continuously differentiable

**central path**: $\{ x^*(t) \mid t > 0 \}$ where $x^*(t)$ solves

\[
\begin{align*}
\text{minimize} & \quad t f_0(x) + \phi(x) \\
\text{subject to} & \quad Ax = b
\end{align*}
\]
Dual points on central path

\[ x = x^*(t) \] if there exists \( w \in \mathbb{R}^p \),

\[ t \nabla f_0(x) + \sum_{i=1}^{m} Df_i(x)^T \nabla \psi_i(-f_i(x)) + A^T w = 0 \]

\((Df_i(x) \in \mathbb{R}^{k_i \times n} \) is derivative matrix of \( f_i)\)

- therefore, \( x^*(t) \) minimizes Lagrangian \( L(x, \lambda^*(t), \nu^*(t)) \), where

\[ \lambda_i^*(t) = \frac{1}{t} \nabla \psi_i(-f_i(x^*(t))), \quad \nu^*(t) = \frac{w}{t} \]

- from properties of \( \psi_i: \lambda_i^*(t) \succ_{K_i^*} 0 \), with duality gap

\[ f_0(x^*(t)) - g(\lambda^*(t), \nu^*(t)) = (1/t) \sum_{i=1}^{m} \theta_i \]
example: semidefinite programming (with $F_i \in S^p$)

minimize $c^T x$
subject to $F(x) = \sum_{i=1}^n x_i F_i + G \preceq 0$

• logarithmic barrier: $\phi(x) = \log \det(-F(x)^{-1})$
• central path: $x^*(t)$ minimizes $tc^T x - \log \det(-F(x))$; hence

$$tc_i - \text{tr}(F_i F(x^*(t))^{-1}) = 0, \quad i = 1, \ldots, n$$

• dual point on central path: $Z^*(t) = -(1/t)F(x^*(t))^{-1}$ is feasible for

maximize $\text{tr}(GZ)$
subject to $\text{tr}(F_i Z) + c_i = 0, \quad i = 1, \ldots, n$
$Z \succeq 0$

• duality gap on central path: $c^T x^*(t) - \text{tr}(GZ^*(t)) = p/t$
Barrier method

**given** strictly feasible $x$, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$.

**repeat**

1. **Centering step.** Compute $x^*(t)$ by minimizing $tf_0 + \phi$, subject to $Ax = b$.
2. **Update.** $x := x^*(t)$.
3. **Stopping criterion.** **quit** if $(\sum_i \theta_i)/t < \epsilon$.
4. **Increase $t$.** $t := \mu t$.

- only difference is duality gap $m/t$ on central path is replaced by $\sum_i \theta_i/t$
- number of outer iterations:

$$\left\lfloor \frac{\log((\sum_i \theta_i)/(\epsilon t^{(0)}))}{\log \mu} \right\rfloor$$

- complexity analysis via self-concordance applies to SDP, SOCP
Examples

**second-order cone program** (50 variables, 50 SOC constraints in $\mathbb{R}^6$)

**semidefinite program** (100 variables, LMI constraint in $\mathbb{S}^{100}$)
family of SDPs \((A \in \mathbb{S}^n, x \in \mathbb{R}^n)\)

\[
\begin{align*}
\text{minimize} & \quad 1^T x \\
\text{subject to} & \quad A + \text{diag}(x) \succeq 0
\end{align*}
\]

\(n = 10, \ldots, 1000\), for each \(n\) solve 100 randomly generated instances
Primal-dual interior-point methods

more efficient than barrier method when high accuracy is needed

• update primal and dual variables at each iteration; no distinction between inner and outer iterations

• often exhibit superlinear asymptotic convergence

• search directions can be interpreted as Newton directions for modified KKT conditions

• can start at infeasible points

• cost per iteration same as barrier method