1 (a) $\quad u_{11} = a_1, \quad u_{12} = c_1$

$\ell_{21} = b_1/u_{11}, \quad u_{22} = a_2 - \ell_{21}u_{12}, \quad u_{23} = c_2$

$\ell_{32} = b_2/u_{22}, \quad u_{33} = a_3 - \ell_{32}u_{23}$

(b) Let $\quad A = LU \quad$ where

$$\underbrace{\begin{bmatrix} a_1 & c_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & c_{m-1} \\ 0 & & b_{m-1} & a_m \end{bmatrix}}_{=A} = \underbrace{\begin{bmatrix} 1 & & & \\ \ell_{21} & \ddots & & 0 \\ & \ddots & 1 & \\ 0 & & \ell_{m,m-1} & 1 \end{bmatrix}}_{=L} \underbrace{\begin{bmatrix} u_{11} & u_{12} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & u_{m-1,m} \\ 0 & & & u_{mm} \end{bmatrix}}_{=U}$$

Algorithm:

$u_{1,1} = a_1, \quad u_{1,2} = c_1$

for $i = 2, \cdots, m-1$

$\quad \ell_{i,i-1} = b_{i-1}/u_{i-1,i-1}$

$\quad u_{i,i} = a_i - \ell_{i,i-1}u_{i-1,i}$

$\quad u_{i,i+1} = c_i$

END

$\ell_{m,m-1} = b_{m-1}/u_{m-1,m-1}, \qquad u_{m,m} = a_m - \ell_{m,m-1}u_{m-1,m}$

(c) In the loops $(m-2)$, we have

$1\,(/), \quad 1\,(\cdot), \quad 1\,(-)$,

thus $\quad (m-2)\cdot 3$.

Total flops: $\quad (m-2)\cdot 3 + 3 = 3m - 3 \simeq 3m$

Standard Gauss Elimination for a dense A

$\sim \frac{2}{3}m^3$ flops

(d) $Ax = b$ is solved by

(1) $Ly = b$

$$\begin{bmatrix} \ddots & & 0 \\ l_{i,i-1} & \ddots & \\ 0 & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ y_i \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ b_i \\ \vdots \end{bmatrix}$$

(2) $Ux = y$

$$\begin{bmatrix} \ddots & & 0 \\ & u_{i,i} \; u_{i,i+1} & \\ 0 & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_i \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ y_i \\ \vdots \end{bmatrix}$$

In (1), Algorithm is

$y_1 = b_1$

For $i = 2, \cdots m$

$\quad y_i = b_i - l_{i,i-1} y_{i-1}$

END

$$\text{flops} = (m-1)\cdot 2 \simeq 2m = O(m)$$

comparing to flops $\simeq \frac{2}{3}m^2$ if L is dense

In (2)

$x_m = y_m / u_{m,m}$

For $i = m-1, \cdots, 1$

$\quad x_i = \frac{1}{u_{i,i}}(y_i - u_{i,i+1} x_{i+1})$

END

$$\text{flops} = (m-1)\cdot 3 + 1 \simeq 3m = O(m)$$

comparing to flops $\simeq m^2$ if U is dense.

2 ($\Rightarrow$) We show first that

$$\lim_{n\to\infty} \|A^n\| = 0 \quad \Rightarrow \quad \rho(A) < 1.$$

Let (λ, v) be an eigenpair for A.

Then $$A v = \lambda v$$

and $$A^n v = \lambda^n v \qquad \forall\, n = 1, 2, \ldots$$

Therefore

$$|\lambda|^n \|v\| = \|\lambda^n v\| = \|A^n v\| \leq \|A^n\| \, \|v\|$$

Since $v \neq 0$ for each λ. we have

$$0 \leq |\lambda|^n \leq \|A^n\|, \qquad \forall\, n = 1, 2.$$

Then $\lim_{n\to\infty} \|A^n\| = 0$ implies $\lim_{n\to\infty} |\lambda|^n = 0$

which in turn implies

$$|\lambda| < 1, \qquad \text{for each ew } \lambda.$$

Therefore $\rho(A) < 1$

($\Leftarrow$) We now show that

$$\lim_{n \to \infty} \|A^n\| = 0 \Leftarrow \rho(A) < 1$$

Let

$$A = Q T Q^*$$

be a Schur factorization of A

where Q is unitary

and T is upper-triangular with the ews of A as its diagonals:

$$T = \begin{bmatrix} \lambda_1 & & X \\ & \ddots & \\ O & & \lambda_m \end{bmatrix}$$

where $\{\lambda_i\}_{i=1}^m$ are ews of A.

We assume that $\rho(A) < 1$. i.e

$$|\lambda_i| < 1, \quad \forall i = 1, \cdots, m.$$

It is easy to show that

$$A^n = Q T^n Q^*, \quad \forall n = 1, 2, \cdots$$

Observe that

$$T = \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_m \end{bmatrix} + \begin{bmatrix} 0 & & X \\ & \ddots & \\ O & & 0 \end{bmatrix}$$

λ_1 as factor for 1^{st} row

Then

$$T^2 = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_m^2 \end{bmatrix} + \begin{bmatrix} 0 & \lambda_1 \otimes & \cdots & \\ & 0 & \lambda_i \otimes & \cdots \\ & & \ddots & \lambda_{m-1} \otimes \\ 0 & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & X \\ & \ddots & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

$$T^3 = \begin{bmatrix} \lambda_1^3 & & 0 \\ & \ddots & \\ 0 & & \lambda_m^3 \end{bmatrix} + \begin{bmatrix} 0 & \lambda_1^2 \otimes & \cdots & \\ & 0 & \lambda_i^2 \otimes & \cdots \\ & & \ddots & \lambda_{m-1}^2 \\ 0 & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \lambda_1 \otimes & \cdots \\ & 0 & 0 & \lambda_i \otimes \\ & & \ddots & \lambda_{m-2} \\ 0 & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & X \\ & \ddots & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

Therefore, $\forall\ k \geq m$

λ_1^{k-1} as a factor for (1-2) elm.

$$T^k = \begin{bmatrix} \lambda_1^k, & \lambda_1^{k-1} \odot, & \lambda_1^{k-2} \odot & \cdots & \lambda_1^{k-i+1} \odot & \cdots & \lambda_1^{k-m} \odot \\ & \lambda_2^k & \lambda_2^{k-1} \odot & \cdots & \lambda_2^{k-i+2} \odot & \cdots & \lambda_2^{k-m+} \odot \\ & & \ddots & & \vdots & & \\ & & & & \lambda_i^k & \cdots & \lambda_i^{k-m+i} \odot \\ & & & & & \ddots & \\ & 0 & & & & & \lambda_m^k \end{bmatrix}$$

Thus, if $\rho(A)<1$, then $|\lambda_i|<1$, $\forall i$, so that

$$\lim_{n\to\infty} T^n = 0$$

Therefore

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} Q\, T^n Q^* = Q\left(\lim_{n\to\infty} T^n\right) Q^*$$

$$= 0$$

Then

$$\lim_{n\to\infty} \|A^n\| = 0 .$$

3 (a). ~~Assume that~~ $\forall k = 0, 1, 2, \ldots$

$$e_{k+1} \le c\, e_k, \quad \text{for } 0 < c < 1.$$

Then

$$e_k \le c\, e_{k-1} \le c^2 e_{k-2}$$
$$\le \cdots$$
$$\le c^k e_0.$$

Therefore

$$(*) \qquad e_k \le c^k e_0, \quad \forall k = 0, \ldots$$

Let n be the number of iterations required for an given accuracy $\varepsilon_{machine}$, i.e.

$$e_n \le \varepsilon_{machine}$$

it follows from $(*)$ that

$$c^n e_0 \le \varepsilon_{machine}$$

Taking log.

$$\log e_0 + n \log c \le \log \varepsilon_{machine}$$

Rearranging it: gives

$$n \ge \frac{(-\log \varepsilon_{machine} + \log e_0)}{(-\log c)}$$

Therefore the nb of iteration required is

$$n \simeq \frac{1}{|\log C|} \, |\log \varepsilon_{machine}|$$

$$\simeq O(|\log \varepsilon_{machine}|)$$

The work requirement is

$$n \cdot O(1) \simeq \frac{1}{|\log C|} \, |\log \varepsilon_{machine}| \cdot O(1)$$

$$\simeq O(|\log \varepsilon_{machine}|)$$

The constant C enters into the estimate as the factor $\frac{1}{|\log C|}$ implying that an "increase" of the work is expected as C closes to 1

3 (b) Assume that $\forall k = 0, 1, 2, \ldots$ ①

$$e_{k+1} \le c\,(e_k)^{\alpha}, \quad \text{for } \alpha > 1, \quad c > 0.$$

Set
$$f_k = \underset{(\ll 1)}{c^{\frac{1}{\alpha-1}} e_k}, \quad \forall k = 0, 1, \ldots$$

Then
$$f_{k+1} \le f_k^{\alpha}, \quad \forall k = 0, 1, \ldots$$

Therefore
$$(*) \qquad f_k \le f_0^{\alpha^k}, \quad \forall k = 0, 1, \ldots$$

Let n be the number of iterations needed for accuracy $\varepsilon_{machine}$, i.e.

$$e_n \le \varepsilon_{machine}.$$

which implies
$$f_n \le c^{\frac{1}{\alpha-1}} \varepsilon_{machine}.$$

We have from $(*)$

$$f_0^{\alpha^n} \le \underset{(\ll 1)}{c^{\frac{1}{\alpha-1}} \varepsilon_{machine}}$$

Taking log,

$$(\log f_0)\, \alpha^n \le \underset{(<0)}{\frac{1}{\alpha-1} \log c + \log \varepsilon_{machine}}$$

Note that

$$\log f_0 = \underset{<0}{\frac{1}{\alpha-1} \log c + \log e_0}$$

Then

$$\alpha^n \geq \frac{-\log \varepsilon_{machine} - \frac{1}{\alpha-1}\log C}{-\log e_0 - \frac{1}{\alpha-1}\log C}$$

Taking log again.

$$(\log\alpha)\, n \geq \log\left(-\log \varepsilon_{machine} - \frac{1}{\alpha-1}\log C\right) - \log\left(-\log e_0 - \frac{1}{\alpha-1}\log C\right)$$

i.e

$$n \geq \frac{1}{\log\alpha}\left[\log\left(-\log \varepsilon_{machine} - \frac{1}{\alpha-1}\log C\right) - \log\left(-\log e_0 - \frac{1}{\alpha-1}\log C\right)\right]$$

Therefore the nb of iterations required is

$$n \approx \frac{1}{\log\alpha} \log(|\log \varepsilon_{machine}|)$$
$$\simeq O\left(\log(|\log \varepsilon_{machine}|)\right)$$

The work requirements is

$$n \cdot O(1) \simeq O\left(\log(|\log \varepsilon_{machine}|)\right)$$
$$\simeq \frac{1}{\log\alpha} \log(|\log \varepsilon_{machine}|)$$

The exponent α enters into the estimate as the constant factor, $\frac{1}{\log\alpha}$, implying that an "increase" of the work is expected as α closes to 1.

3 (4) Alternative Solution:

Assume that $\forall k = 0, 1, 2, \dots$

$$e_{k+1} \le c\,(e_k)^{\alpha}, \quad \text{for } \alpha > 1, \quad c > 0.$$

Then

$$e_k \le c\,(e_{k-1})^{\alpha} \le c^{1+\alpha} e_{k-2}^{\alpha^2}$$

$$\le c^{1+\alpha+\alpha^2} e_{k-3}^{\alpha^3} \le \cdots \le \cdots$$

$$\le c^{1+\alpha+\cdots\alpha^{k-1}} e_0^{\alpha^k} = c^{\frac{\alpha^k - 1}{\alpha - 1}} e_0^{\alpha^k}$$

Therefore

$$(**) \qquad e_k \le c^{\frac{\alpha^k-1}{\alpha-1}} e_0^{\alpha^k}, \quad \forall k = 0, \dots$$

Let n be the nb of iterations required for a given accuracy $\varepsilon_{machine}$, i.e

$$e_n \le \varepsilon_{machine}$$

We have from $(**)$

$$c^{\frac{\alpha^n-1}{\alpha-1}} e_0^{\alpha^n} \le \varepsilon_{machine}$$

Taking log.

$$\frac{\alpha^n - 1}{\alpha - 1} \log c + (\log e_0)\, \alpha^n \le \log \varepsilon_{machine}$$

Rearrange

$$\left(-\log e_0 - \frac{1}{\alpha-1}\log c\right)\alpha^n \ge -\log \varepsilon_{machine} - \frac{1}{\alpha-1}\log c$$