NUMERICAL ANALYSIS

Third Test — Math 4364 (Fall 2011) — October 11, 2011

> This exam has 4 questions, for a total of 100 points.
> Please answer the questions in the spaces provided on the question sheets.
> If you run out of room for an answer, continue on the back of the page.

Name and ID: Solution Keys

[25 points] 1. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Find the eigenvalues and associated eigenvectors of the matrix A.

1/ Find λ s.t. $\det(A - \lambda I) = 0$

i.e $\det\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} \left(= \begin{array}{ccccc} 1-\lambda & 1 & 1 & 1-\lambda & 1 \\ 1 & 1-\lambda & 0 & 1 & 1-\lambda \\ 1 & 0 & 1-\lambda & 1 & 0 \end{array} \right)$

5/ $= (1-\lambda)^3 + 0 + 0 - (1-\lambda) - (1-\lambda) - 0$

$= (1-\lambda)\left[(1-\lambda)^2 - 2\right] = (1-\lambda)(1-\sqrt{2}-\lambda)(1+\sqrt{2}-\lambda)$

$= 0$

5/ $\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 1+\sqrt{2}, \quad \lambda_3 = 1-\sqrt{2}$

2/ Find $v \neq 0$ s.t. $(A - \lambda I)v = 0$. $\quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

5/ $\lambda = 1: \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{matrix} y+z=0 \\ x=0 \end{matrix} \Leftrightarrow \begin{matrix} x=0 \\ y=-z \end{matrix}$

$\Rightarrow V_{\lambda=1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

5/ $\lambda = 1+\sqrt{2} \Rightarrow \begin{bmatrix} -\sqrt{2} & 1 & 1 \\ 1 & -\sqrt{2} & 0 \\ 1 & 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{matrix} x = \sqrt{2}y \\ x = \sqrt{2}z \\ \Rightarrow y = z \end{matrix} \Rightarrow V_{\lambda=1+\sqrt{2}} = \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix}$

5/ $\lambda = 1-\sqrt{2} \Rightarrow \begin{bmatrix} \sqrt{2} & 1 & 1 \\ 1 & \sqrt{2} & 0 \\ 1 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{matrix} x = -\sqrt{2}y \\ x = -\sqrt{2}z \\ \Rightarrow y = z \end{matrix} \Rightarrow V_{\lambda=1-\sqrt{2}} = \begin{bmatrix} -\sqrt{2} \\ 1 \\ 1 \end{bmatrix}$

15 points 2. Determine if the matrix A is diagonalizable and, if so, find P and D with $A = PDP^{-1}$.

5/ 1°/ $A = A^T$. Thus A is diagonalizable

2°/ From the result of Pb 1, we have

5/ $$P = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

5/ $$D = \begin{bmatrix} 1+\sqrt{2} & 0 & 0 \\ 0 & 1-\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

30 points 3. Find the first two iterations obtained by the Power method applied to the matrix A.

Since $A = A^T$ (symmetric), we apply the Power method using $x^0 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$:

$$x^{(0)} = x^0 / \|x^0\|_2$$

5/

For $k = 1, 2$

$$y = A x^{(k-1)}$$

$$\mu^{(k)} = y^T x$$

$$x^{(k)} = y / \|y\|_2$$

$x^0 = (-1, 0, 1)^T$, $\|x^0\|_2 = \sqrt{2}$

5/ $x^0 = x^0/\|x^0\|_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$k = 1$,

$$y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

5/ $\mu^{(1)} = \frac{1}{\sqrt{2}}(0, -1, 0) \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0$

$\|y\|_2 = \frac{1}{\sqrt{2}}$

5/ $x^{(1)} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$k = 2$

$$y = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \quad \|y\|_2 = \sqrt{2}$$

5/ $\mu^{(2)} = (-1, -1, 0) \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = 1$

5/ $x^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

$k = 3$.

$$y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}, \quad \|y\|_2 = \frac{3}{\sqrt{2}}$$

$$\mu^{(3)} = \frac{1}{\sqrt{2}}(-2, -2, -1) \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = 2$$

$$x^{(3)} = \frac{1}{3} \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

30 points 4. Use Householder's method to place the matrix A in tridiagonal form.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ccc} 1 & 2 & 3 \end{array}$$

1 5/ $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\|x\|_2 = \sqrt{2}$, $\text{sign}(x_1) = 1$, $v = x + \text{sign}(x_1)\|x\|_2 e_1 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$

$v^T v = 2(2+\sqrt{2}) = 2\sqrt{2}(1+\sqrt{2})$

5/ The Householder's reflector is

$$H = I - 2\frac{vv^T}{v^Tv} = I - \frac{1}{\sqrt{2}\cdot(1+\sqrt{2})} vv^T$$

i.e.

$$\forall z \in \mathbb{R}^2, \quad Hz = z - \frac{1}{\sqrt{2}(1+\sqrt{2})}(v^Tz)\, v$$

2 5/ $z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Hz = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}(1+\sqrt{2})}(1+\sqrt{2})\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \simeq \begin{pmatrix} -0.7071 \\ -0.7071 \end{pmatrix}$

$v^Tz = 1+\sqrt{2}$

3 5/ $z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $Hz = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}(1+\sqrt{2})}\cdot 1 \cdot \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \simeq \begin{pmatrix} -0.7071 \\ +0.7071 \end{pmatrix}$

$v^Tz = 1$

Let $Q = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & H \end{array}\right)$. Then

$$QA = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \end{pmatrix} \begin{array}{c} 1 \\ 4 \\ 5 \end{array} \\ \begin{array}{ccc} H1 & H2 & H3 \end{array} \end{array}$$

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.

④ $z = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, $Hz = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} - \frac{1}{\sqrt{2}(1+\sqrt{2})} \left(-(1+\sqrt{2})\right) \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

5/

$v^T z = -(1+\sqrt{2})$

⑤ $z = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{pmatrix}$, $Hz = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{pmatrix} - \frac{1}{\sqrt{2}(1+\sqrt{2})} (-1) \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

5/

$v^T z = -1$

There fore

$$Q A Q^T = \begin{array}{c} \begin{pmatrix} 1 & -\sqrt{2} & 0 \\ -\sqrt{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} H① \\ H④ \\ H⑤ \end{array}$$