NUMERICAL ANALYSIS II

First Exam — **Math 6371-13999 (Spring 2011)** — **March 24, 2011**

This exam has 3 questions, for a total of 100 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Name and ID: Solutions

30 points 1. Determine the interpolating polynomial of degree 2 in both the Lagrange and Newton forms for the function $f(x) = \frac{2}{1+x^2}$ using the interpolation points $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.

(15pts) Lagrange:

$$p(x) = f_0 \ell_0(x) + f_1 \ell_1(x) + f_2 \ell_2(x)$$

where $f_0 = f(x_0) = 1, \quad f_1 = f(x_1) = 2, \quad f_2 = f(x_2) = 1$

$$\ell_0(x) = \frac{1}{2}x(x-1), \quad \ell_1(x) = -(x+1)(x-1), \quad \ell_2(x) = \frac{1}{2}x(x+1)$$

Therefore

$$p(x) = 2 - x^2$$

(15pts) Newton.

$$p(x) = c_0 \varphi_0(x) + c_1 \varphi_1(x) + c_2 \varphi_2(x)$$

where $c_0 = [x_0]f = 1, \quad c_1 = [x_0, x_1]f = 1, \quad c_2 = [x_0, x_1, x_2]f = -1$

$$\varphi_0(x) = 1, \quad \varphi_1(x) = x+1, \quad \varphi_2(x) = x(x+1)$$

Therefore

$$p(x) = 2 - x^2$$

Divided-difference table

-1	1		
		1	
0	2		-1
		-1	
1	1		

$\boxed{\text{30 points}}$ 2. Let x_0, x_1, x_2 and x_3 be 4 equally-spaced nodes with steplength $h > 0$, i.e., $x_i = x_0 + ih$, $i = 1, 2, 3$. Let $f(x)$ be a function defined on $[x_0, x_3]$. Assume that $f_i = f(x_i)$ is known at x_0, x_1, and x_2, but is unknown at x_3.

Integrate the polynomial which interpolates $f(x)$ at x_0, x_1 and x_2 to determine the coefficients w_i, $i = 0, 1, 2$, of the associated quadrature formula approximating the integration of $f(x)$ over $[x_2, x_3]$.

$$\int_{x_2}^{x_3} f(x)dx \approx \sum_{i=0}^{2} w_i f(x_i)$$

Remark: Formula of this type are used in the so-called "explicit multi-step methods" for solving ordinary differential equations.

- Lagrange: $p(x) = \sum_{i=0}^{2} f(x_i)\, l_i(x)$, $\quad l_i(x) = \prod_{\substack{j\neq i\\ j=0}}^{2} \frac{x - x_j}{x_i - x_j}$
- Quadrature: $\int_{x_2}^{x_3} f(x)dx \simeq \int_{x_2}^{x_3} p(x)dx = \sum_{i=0}^{2} f(x_i)\int_{x_2}^{x_3} l_i(x)dx = \sum_{i=0}^{2} w_i f(x_i)$

  where the coefficients are $w_i = \int_{x_2}^{x_3} l_i(x)dx$, $i = 0, 1, 2$.
- Let $x = x_0 + th$, $dx = h\,dt$. $\quad \tilde{l}_i(t) = l_i(x) = \prod_{\substack{j\neq i\\ j=0}}^{2} \frac{x - x_j}{x_i - x_j} = \prod_{\substack{j\neq i\\ j=0}}^{2} \frac{t - j}{i - j}$.
- Then $w_i = h\int_2^3 \tilde{l}_i(t)\,dt$, $i = 0, 1, 2$.
- $\Rightarrow$

$$\tilde{l}_0(t) = \frac{1}{2}(t-1)(t-2), \quad w_0 = \frac{1}{2}h\int_2^3 (t-1)(t-2)dt = \frac{1}{2}h\int_0^1 t(t+1)\,dt$$
$$= \frac{1}{2}h\left[\frac{1}{3}t^3 + \frac{1}{2}t^2\right]_0^1 = \frac{5}{12}h$$

$$\tilde{l}_1(t) = -t(t-2), \quad w_1 = -h\int_2^3 t(t-2)dt = -h\int_0^1 t(t+2)\,dt$$
$$= -h\left[\frac{1}{3}t^3 + t^2\right]_0^1 = -\frac{4}{3}h$$

$$\tilde{l}_2(t) = \frac{1}{2}t(t-1), \quad w_2 = \frac{1}{2}h\int_2^3 t(t-1)\,dt = \frac{1}{2}h\left[\frac{1}{3}t^3 - \frac{1}{2}t^2\right]_2^3$$
$$= \frac{23}{12}h$$

- Therefore

$$\int_{x_2}^{x_3} f(x)\,dx \simeq \frac{h}{12}\Big(5f(x_0) - 16f(x_1) + 23f(x_2)\Big).$$

Check the exactness for $f(x) = 1, x, x^2$ (with $x_1 = 0$)
✓ ✓ ✓

40 points 3. Suppose we use the (composite) trapezoidal rule to approximate the definite integral $I(f) = \int_0^1 x^5 dx$. (Note that $I(f) = \frac{1}{6}$.)

(10 pts) a) Compute the sequence of trapezoidal sums $T(h)f$ for the step lengths $h_1 = 1$, $h_2 = \frac{1}{2}$, $h_3 = \frac{1}{4}$.

(5 pts) b) Apply the Euler-Maclaurin formula to show

$$T(h)f = \frac{1}{6} + \frac{5}{12}h^2 - \frac{1}{12}h^4$$

(5 pts) c) Use the result in b) to check your results in a). What is the order of accuracy of your results in a).

(5 pts) d) For some classes of function the composite trapezoidal rule exhibits so-called *super-convergence*. What is meant by this term? Give an example of a class of functions for which this is true.

(10 pts) e) Apply Romberg's method to the trapezoidal sums computed in a) to compute the values of items $T_{m,k}$ for $m = 1, 2, 3$ and $1 \le k \le m$.

(5 pts) f) Apply the error bound for Romberg's method to show

$$T_{3,3} = I(f)$$

a) $h_1 = 1$, $T(h_1)f = h_1 \frac{1}{2}(f(0) + f(1)) = \frac{1}{2}$

$h_2 = \frac{1}{2}$, $T(h_2)f = h_2 \left[\frac{1}{2}(f(0)+f(1)) + f(\frac{1}{2})\right] = \frac{17}{64}$

$h_3 = \frac{1}{4}$, $T(h_3)f = h_3 \left[\frac{1}{2}(f(0)+f(1)) + f(\frac{1}{4}) + f(\frac{1}{2}) + f(\frac{3}{4})\right] = \frac{197}{1024}$

b) By Euler-Maclaurin Formula

$$T(h)f = I(f) + \frac{h^2}{12}(f'(1) - f'(0)) - \frac{h^4}{720}(f'''(1) - f'''(0))$$

$$= \frac{1}{6} + \frac{5}{12}h^2 - \frac{1}{12}h^4$$

c) By the result in b), $T(h_1)f = \frac{1}{2}$, $T(h_2)f = \frac{17}{64}$, $T(h_3)f = \frac{197}{1027}$.
same as in a).

We have $T(h)f - I(h) = \frac{5}{12}h^2 - \frac{1}{12}h^4 = O(h^2)$

Thus the order of accuracy of the results in a) is 2.

d) For function $f \in C^\infty$ and periodic on $[a, b]$, the trapezoidal rule exhibit superconvergence, meaning that the error $\to 0$ faster than any power of h as $h \to 0$. Example, trigonometric polynomial of degree m, the trapezoidal rule with $h = 2\pi/n$, $n \ge m+1$ integrates it exactly.

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.

e) By Romberg's method,

$T_{11} = T(h_1)f = \frac{1}{2}$, $T_{21} = T(h_2)f = \frac{17}{64}$, $T_{31} = T(h_3)f = \frac{197}{1024}$

$$T_{22} = T_{21} + \frac{T_{21} - T_{11}}{4-1} = \frac{3}{16}, \quad T_{32} = T_{31} + \frac{T_{31} - T_{21}}{4-1} = \frac{43}{256}$$

$$T_{33} = T_{32} + \frac{T_{32} - T_{22}}{4-1} = \frac{1}{6}$$

m	T_{m1}	T_{m2}	T_{m3}
1	$\frac{1}{2}$		
2	$\frac{17}{64}$	$\frac{3}{16}$	
3	$\frac{197}{1024}$	$\frac{43}{256}$	$\frac{1}{6}$

f) By the error bound for Romberg's method (Theorem 5.2.1)

$$T_{m,k} - I(f) = r_k h^{2k} (b-a) f^{(2k)}(\xi), \quad \xi \in (0,1).$$

Since $f(x) = x^5$, $f^{(6)}(\xi) = 0$, $\forall \xi \in (0,1)$,

then $T_{33} - I(f) = 0$

i.e. $T_{33} = I(f)$ exactly.