NUMERICAL ANALYSIS II

Second Exam   Math 6371-13999 (Spring 2011)   May 5, 2011

This exam has 3 questions, for a total of 100 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Name ~~and ID~~: Solutions

35 points 1. Consider Heun's method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right].$$

(a) Show that Heun's method is an explicit two-stage RK method.

(b) Prove that Heun's method has order 2 with respect to h.

(a) (20pts) Heun's method can be reformulated as

$$\xi_1 = y_n$$

$$\xi_2 = y_n + h\, a_{2,1} f(t_n, \xi_1)$$

$$y_{n+1} = y_n + h\left[b_1 f(t_n + c_1 h, \xi_1) + b_2 f(t_n + c_2 h, \xi_2)\right]$$

which is an explicit two-stage RK method with the RK tableau

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

i.e, $c_1 = 0$, $c_2 = 1$, $a_{2,1} = 1$, $b_1 = b_2 = \frac{1}{2}$

(b) (15pts) The above RK tableau satisfies the condition for order $p = 2$, (3.7) page 39,

$$b_1 + b_2 = 1, \quad b_2 c_2 = \frac{1}{2}, \quad a_{2,1} = c_2.$$

Thus, Heun's method has order $p = 2$ w.r.t. h.

$\boxed{35 \text{ points}}$ 2. Consider the theta method

$$y_{n+1} = y_n + h\left[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\right].$$

(a) Prove that the theta method is convergent for every $\theta \in [0, 1]$.

(b) Prove that the theta method is A-stable if and only if $\theta \in [0, \frac{1}{2}]$.

(a) (30 pts) We follow the proofs of Theorems 1.1 and 1.2.

From Page 14, we have

$$e_{n+1} = e_n + \theta h\left[f(t_n, y(t_n) + e_n) - f(t_n, y(t_n))\right] + (1-\theta)h\left[f(t_{n+1}, y(t_{n+1}) + e_{n+1}) - f(t_{n+1}, y(t_{n+1}))\right] + O(h^{\alpha+1})$$

where $\alpha = \begin{cases} 2, & \theta = \frac{1}{2} \\ 1 & \theta \neq \frac{1}{2} \end{cases}$

It follows from the Lipschitz condition and the triangle inequality that, for $\theta \in [0, 1]$,

$$\|e_{n+1}\| \le \|e_n\| + \theta h\lambda \|e_n\| + (1-\theta) h\lambda \|e_{n+1}\| + c h^{\alpha+1}$$

Assume that for small h, $(1-\theta)h\lambda < 1$, we have

$$\|e_{n+1}\| \le \left(\frac{1+\theta h\lambda}{1-(1-\theta)h\lambda}\right)\|e_n\| + \left(\frac{c}{1-(1-\theta)h\lambda}\right) h^{\alpha+1}$$

By induction on n, we deduce that

$$\|e_n\| \le \frac{c}{\lambda}\left[\left(\frac{1+\theta h\lambda}{1-(1-\theta)h\lambda}\right)^n - 1\right] h^\alpha$$

Since $\dfrac{1+\theta h\lambda}{1-(1-\theta)h\lambda} = 1 + \dfrac{h\lambda}{1-(1-\theta)h\lambda} \le \exp\left(\dfrac{h\lambda}{1-(1-\theta)h\lambda}\right)$,

We have

$$\|e_n\| \le \frac{c}{\lambda} h^\alpha \left(\frac{1+\theta h\lambda}{1-(1-\theta)h\lambda}\right)^n \le \frac{c}{\lambda} h^\alpha \exp\left(\frac{nh\lambda}{1-(1-\theta)h\lambda}\right)$$

For $\forall n$ s.t. $nh < t^*$,

$$\|e_n\| \le \frac{c}{\lambda} h^\alpha \exp\left(\frac{t^*\lambda}{1-(1-\theta)h\lambda}\right)$$

Thus $\lim\limits_{\substack{h\to 0 \\ 0 \le nh \le t^*}} \|e_n\| = 0$, i.e, the theta method is convergent for every $\theta \in [0, 1]$.

(b) (25pts) Appling the theta method to the linear equation (4.7), i.e.

$$y' = \lambda y, \quad t \geq 0; \quad y(0) = 1$$

produces a geometric solution sequence

$$y_n = [r_\theta(h\lambda)]^n, \quad n = 0, 1, \cdots$$

where

$$r_\theta(z) = \frac{1 + \theta z}{1 - (1-\theta) z}; \qquad \forall \theta \in [0, 1].$$

Then the linear stability domain for the theta method is.

$$D_\theta = \{ z \in \mathbb{C}, \ |r_\theta(z)| < 1 \}$$

To check A-stability, i.e,

$$\mathbb{C}^{-} = \{ z \in \mathbb{C} : \operatorname{Re} z < 0 \} \subseteq D$$

We represent $\mathbb{C}^{-} = \{ z = \rho e^{i\alpha}, \quad \rho > 0, \quad \alpha \in (\frac{1}{2}\pi, \frac{3}{2}\pi) \}$ in polar coordinates.

We look for θ s.t. $|r_\theta(\rho e^{i\alpha})| < 1$.

It would be equivalent to

$$|1 + \theta \rho e^{i\alpha}|^2 < |1 - (1-\theta) \rho e^{i\alpha}|^2$$

$\Longleftrightarrow$

$$1 + 2\theta \rho \cos\alpha + \theta^2 \rho^2 < 1 - 2(1-\theta) \rho \cos\alpha + (1-\theta)^2 \rho^2$$

$\Longleftrightarrow$ $$(1 - 2\theta) \rho > 2 \cos\alpha$$

Since $\cos\alpha < 0$ for $\forall \alpha \in (\frac{1}{2}\pi, \frac{3}{2}\pi)$, i.e, $\forall z \in \mathbb{C}^{-}$, the theta method is therefore A-stable if and only if $\quad 1 - 2\theta \geq 0$, i.e $\quad \theta \in [0, \frac{1}{2}]$.

~~30~~ points (10 pts) 3. Prove that the Gauss-Seidel iteration converges whenever the matrix A is symmetric and positive definite.

The GS iteration, applied to $Ax = b$, can be viewed as a regular splitting iteration

$$P x^{[k+1]} = N x^{[k]} + b, \quad k = 0, 1, \dots$$

with

$$P = D - L_0, \quad N = U_0$$

where we split A as

$$A = (D - L_0) - U_0$$

Since A is symmetric, positive definite, Theorem 12.2 and the positive definiteness of the matrix $Q := P + P^T - A$ imply convergenc. For the GS splitting,

$$Q = P + P^T - A = (D - L_0) + (D - L_0)^T - A$$
$$= D - L_0 + D - L_0^T = D + (D - L_0 - U_0) - A$$
$$= D$$

i.e. Q is simply the diagonal matrix of A. Thus the positive definiteness of A implies that

$$Q > 0.$$

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.