1. Write out the cardinal functions \( L_i(x) \) appropriate to the problem of interpolating the following table, and give the Lagrange form of the interpolating polynomial:

\[
\begin{array}{c|ccc}
 x & \frac{1}{3} & \frac{1}{4} & 1 \\
 f(x) & 2 & -1 & 7 \\
\end{array}
\]

**Solution** The cardinal functions \( L_i(x) \) are

\[
L_0(x) = \frac{(x - \frac{1}{4})(x - 1)}{(\frac{1}{3} - \frac{1}{4})(\frac{1}{3} - 1)} = -18(x - \frac{1}{4})(x - 1)
\]

\[
L_1(x) = \frac{(x - \frac{1}{3})(x - 1)}{(\frac{1}{4} - \frac{1}{3})(\frac{1}{4} - 1)} = 16(x - \frac{1}{3})(x - 1)
\]

\[
L_2(x) = \frac{(x - \frac{1}{4})(x - \frac{1}{3})}{(1 - \frac{1}{3})(1 - \frac{1}{4})} = 2(x - \frac{1}{3})(x - \frac{1}{4})
\]

There, the interpolating polynomial in the Lagrange form is

\[
p_2(x) = -36(x - \frac{1}{4})(x - 1) - 16(x - \frac{1}{3})(x - 1) + 14(x - \frac{1}{3})(x - \frac{1}{4})
\]

2. Construct a divided-difference diagram for the function \( f \) given in the following table, and write out the Newton form of the interpolating polynomial

\[
\begin{array}{c|cccc}
 x & 1 & \frac{3}{2} & 0 & 2 \\
 f(x) & 3 & \frac{13}{4} & 3 & \frac{5}{3} \\
\end{array}
\]

**Solution** The complete diagram is

\[
\begin{array}{c|cccc}
 x & f[] & f[\cdot] & f[\cdot,\cdot] & f[\cdot,\cdot,\cdot] \\
 1 & 3 & & & \\
 \frac{3}{2} & \frac{13}{4} & \frac{1}{2} & \frac{1}{3} & -2 \\
 0 & 3 & \frac{1}{6} & \frac{-5}{3} & \frac{-2}{3} \\
 2 & \frac{5}{3} & & & \\
\end{array}
\]

where the first entry in column 3 is

\[
f[x_0, x_1] = \frac{\frac{13}{4} - 3}{\frac{3}{2} - 1} = \frac{1}{2}
\]
and after completion of column 3, the first entry in column 4 is
\[ f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{1}{6} - \frac{1}{2}}{0 - 1} = \frac{1}{3} \]
Thus, the Newton form of the interpolating polynomial is
\[ p_3(x) = 3 + \frac{1}{2}(x - 1) + \frac{1}{3}(x - 1)(x - \frac{3}{2}) - 2(x - 1)(x - \frac{3}{2})x \]

3. Write out the cardinal functions \( H_i(x) \) and \( \hat{H}_i(x) \) appropriate to the problem of interpolating the following table, and give the Hermite interpolating polynomial:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

**Solution**
First compute the Lagrange polynomials \( L_i(x) \) and their derivatives:
\[
L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 1}{0 - 1} = 1 - x, \quad L_0'(x) = -1, \\
L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{1 - 0} = x, \quad L_1'(x) = 1
\]
The cardinal functions \( H_i(x) \) and \( \hat{H}_i(x) \) are
\[
H_0(x) = [1 - 2(x - x_0)L_0'(x_0)]L_0^2(x) = (1 + 2x)(1 - x)^2, \\
\hat{H}_0(x) = (x - x_0)L_0^2(x) = x(1 - x)^2, \\
H_1(x) = [1 - 2(x - x_1)L_1'(x_1)]L_1^2(x) = (3 - 2x)x^2, \\
\hat{H}_1(x) = (x - x_1)L_1^2(x) = (x - 1)x^2
\]
Thus, the Hermite interpolating polynomial is
\[ p_3(x) = 2(1 + 2x)(1 - x)^2 + x(1 - x)^2 + (3 - 2x)x^2 + 2(x - 1)x^2 \]

4. Construct a divided-difference diagram for the function \( f \) given in the following table, and give the Hermite interpolating polynomial:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

**Solution**
The complete diagram is
where the first entry in column 3 is \( f'(x_0) \), the second entry in column 3 is 
\[
\begin{array}{c}
0 \\
0 \\
1 \\
1 \\
\end{array}
\]
and after completion of column 3, the first entry in column 4 is
\[
f[z_0, z_1, z_2] = f[z_1, z_2] - f[z_0, z_1] \\
\frac{z_2 - z_0}{1 - 1} = -2
\]
Thus, the Newton form of the Hermite interpolating polynomial is
\[
p_3(x) = 2 + x - 2x^2 + 5x^2(x - 1)
\]
20 points

5. Determine the parameters \( a, b, c, d, e, f, g, \) and \( h \) so that \( S(x) \) is a natural cubic spline, where
\[
S(x) = \begin{cases} 
  ax^3 + bx^2 + cx + d, & x \in [-1, 0] \\
  ex^3 + fx^2 + gx + h, & x \in [0, 1] 
\end{cases}
\]
with interpolating conditions
\[
S(-1) = 1, \quad S(0) = 2, \quad S(1) = -1
\]

Solution \ From the interpolation conditions, we have
\[
S(0) = 2 \quad \Rightarrow \quad d = 2, \quad h = 2, \\
S(-1) = 1 \quad \Rightarrow \quad -a + b - c + d = 1, \\
S(1) = -1 \quad \Rightarrow \quad e + f + g + h = -1
\]
Since \( S'(x) = \begin{cases} 
3ax^2 + 2bx + c, & x \in [-1, 0] \\
3ex^2 + 2fx + g, & x \in [0, 1] 
\end{cases} \)
we have \( c = g \) from the continuity condition of \( S' \) at \( x = 0 \), i.e.,
\[
S'(0) = c = g.
\]
Also, since
\[
S''(x) = \begin{cases} 
6ax + 2b, & x \in [-1, 0] \\
6ex + 2f, & x \in [0, 1] 
\end{cases}
\]
we have $b = f$ from the continuity condition of $S''$ at $x = 0$, i.e.,

\[ S''(0) = 2b = 2f. \]

In order for $S$ to be a natural cubic spline, we must have

\[
\begin{align*}
S''(-1) &= 0 \quad \Rightarrow \quad -6a + 2b = 0 \\
S''(1) &= 0 \quad \Rightarrow \quad 6e + 2f = 0
\end{align*}
\]

From all of these equations, we obtain

\[
a = -1, \quad b = -3, \quad c = -1, \quad d = 2, \quad e = 1, \quad f = -3, \quad g = -1, \quad h = 2
\]