University of Houston
Department of Mathematics
Open Access Graduate Instruction
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Fall 2010 Online *Live* Course
Applicable Analysis

- **Prerequisites:** Linear Algebra, basic analysis, and advanced curiosity.
- **Course Material:** Function spaces, an introduction to Sobolev spaces $R$, linear operators, the Fredholm alternative, the Fredholm splitting theorem, linear integral equations, linear 2-point boundary value problems, nonlinear operators, fixed point theorems, nonlinear systems of initial value problems, and nonlinear boundary value problems.
In the first session, we reviewed linear algebra.

Last week, we introduced metric spaces, separable metric spaces, open sets, closed sets, bounded sets, convergent sequences, compact sets, Cauchy sequences, and complete metric spaces. We also discussed some examples and proved

\[ \text{Theorem: If } X = C([0,1], R) \text{ and } d(u, v) = \|u - v\|_{\infty} \]

then \((X, d)\) is a complete metric space.
You were also given some exercises (in addition to sending me a photo):

- If $X = C([0,1], R)$ and $d(u, v) = \| u - v \|_2$
  then $(X, d)$ is NOT a complete metric space.

- If $k \in N$, $X = C^k([0,1], R)$ and $d(u, v) = \| u - v \|_\infty$
  then $(X, d)$ is NOT a complete metric space.

- If $k \in N$, $X = C^k([0,1], R)$ and
  $$d(u, v) = \sum_{j=0}^{k} \| u^{(j)} - v^{(j)} \|_\infty$$
  then $(X, d)$ is a complete metric space.

**Note:** $[0,1]$ can be replaced by $[a, b]$ in any of these results.
If $X = C([0,1], \mathbb{R})$ and $d(u, v) = \|u - v\|_2$
then $(X, d)$ is NOT a complete metric space.

So let:

Recall:

$$\|v\|_2 = \left( \int_0^1 v(x)^2 \, dx \right)^{\frac{1}{2}}.$$ 

$$f_n(x) = \begin{cases} 
0, & x \leq \frac{1}{2} - \frac{1}{2n} \\
\frac{1}{2 + n}(x - \frac{1}{2}), & \frac{1}{2} - \frac{1}{2n} < x \leq \frac{1}{2} + \frac{1}{2n} \\
1, & \frac{1}{2} + \frac{1}{2n} < x \leq 1
\end{cases}$$

Then $\{f_n\}$ is a C.S. in

$(X, d)$.

If we assume

$$f \in C([0,1], \mathbb{R}) \ni f_n \to f$$

with $\|v\|_2$, then you can prove $f = 1$ for $x > \frac{1}{2}$ and $f = 0$ for $x < \frac{1}{2}$. **HUGE PROBLEM**

$:.$ There is no such

$$f \in C([0,1], \mathbb{R})$$

$$\implies (X, d) \text{ is not complete.}$$

(Fill in the details.)
If \( k \in \mathbb{N} \), \( X = C^k([0,1], \mathbb{R}) \) and \( d(u, v) = \|u - v\|_{\infty} \), then \((X, d)\) is NOT a complete metric space.

Sol’n: let \( f(x) = |x - \frac{1}{2}| \).

Note: \( f \in C^1([0,1], \mathbb{R}) \).

But, \( f \) a sequence \( \{f_n\} \) of polynomials converging uniformly to \( f \), and each \( f_n \in C^1([0,1], \mathbb{R}) \).

\( \therefore (X, d) \) is not complete.
If \( k \in \mathbb{N} \), \( X = C^k([0,1], \mathbb{R}) \) and
\[
d(u,v) = \sum_{j=0}^{k} \| u^{(j)} - v^{(j)} \|_{\infty},
\]
then \((X,d)\) is a complete metric space.

**Soln:** Let \( \{f_n\} \) be a C.S. in \((X,d)\). Let \( X_0 = \mathcal{C}([0,1], \mathbb{R}) \) and \( d_0(u,v) = \|u-v\|_{\infty} \).
We know \((X_0,d_0)\) is a complete metric space.
Also, \( \{f_n^{(j)}\} \) is a C.S. in \((X_0,d_0)\) for \( j = 0, \ldots, k \).

\[ \therefore \forall F_0, F_1, \ldots, F_k \in X_0 \implies f_n^{(j)} \to F_j \text{ wrt } d_0. \]

We need to prove \( F_0 \in X \) and \( F_0^{(0)} = F_j^{(0)} \).
If we do this, then we will have \( f_n \to F_0 \) wrt \( d \) and \( F_0 \in X \).

\[ \text{Just use the } F_0^{(0)} = F_j^{(0)}. \]

\[ \text{Find: } F_0 \in \mathcal{C}([0,1], \mathbb{R}) \text{ and } F_0' = F_1. \]

Similarly, \( F_j^{(j)} = F_j \forall j \).

\[ \therefore F_j^{(j)} = F_j \forall j. \]
I assume you know the definition of a vector space over a field $F$.

What is a norm on a vector space, and what is a normed linear space?

Let $X$ be a vector space (over $F$).

A norm on $X$ is a function $\|\cdot\| : X \to \mathbb{R}$ such that:

1. $\|x\| \geq 0 \forall x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
2. $\|ax\| = |a| \|x\| \forall a \in F, x \in X$.
3. $\|x+y\| \leq \|x\| + \|y\| \forall x, y \in X$.

Note: If $\|\cdot\|$ is a norm on $X$ then $d(x, y) = \|x - y\|$ is a metric on $X$.

Notation: $(X, \|\cdot\|)$ is a normed linear space.

What is a Banach space?

A complete normed linear space. We know some Banach spaces.

1. $(C([a,b], \mathbb{R}), \|\cdot\|_\infty)$
2. $(C^k([a,b], \mathbb{R}), \|\cdot\|_k)$ with

$$\|u\|_k = \sum_{j=0}^{k} \|u^{(j)}\|_\infty.$$
3. $(\mathbb{R}^n, \| \cdot \|_1)$ where $\| \cdot \|_1$ is any norm.

4. $(X, \| \cdot \|_1)$ where $X$ is finite dim and $X$ is any norm.

All norms are equivalent on a finite dim space. i.e., If $X$ is finite dim, and $\| \cdot \|_1$ and $\| \cdot \|_\ast$ are norms on $X$ then $\exists A, B > 0 \exists A \| x \| \leq \| x \|_\ast \leq B \| x \|$ for all $x \in X$. 
What is a linear transformation?

Suppose $X$ and $Y$ are vector spaces over $F$. A linear transformation from $X$ to $Y$ is a function

$L: X \rightarrow Y$

\( \circ \) \( L(\alpha x) = \alpha L(x), \quad \forall \alpha \in F, \quad \forall x \in X \)

\( \circ \circ \) \( L(x+y) = L(x) + L(y), \quad \forall x, y \in X \).
Define the notion of a bounded linear operator.

\[ s p s e (X, \| \cdot \|_X) \text{ and } (Y, \| \cdot \|_Y). \]

A bounded linear operator from \( X \to Y \) (wrt \( \| \cdot \|_X \) and \( \| \cdot \|_Y \) respectively) is a linear transformation from \( T \) from \( X \to Y \) such that:

\[ \exists K \geq 0 \quad \| T(x) \|_Y \leq K \| x \|_X \]

\( \forall x \in X. \)

\[ \mathcal{L}(X,Y) = \{ T \mid T \text{ is a bounded linear operator from } X \to Y \}. \]

Notes:

1. \( \mathcal{L}(X,Y) \) is a vector space.
2. If \( X = Y \) we write \( \mathcal{L}(X) \).
3. There is a "standard norm" that can be placed on $L(X,Y)$. It is called the operator norm. We define $\|\cdot\| : L(X,Y) \to \mathbb{R}$ via

$$\|T\| = \sup_{x \in X, \|x\| \leq 1} \|T(x)\|_Y.$$  

Q: Is every linear transformation from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$ an element of $L(X,Y)$?

A: if and only if $X$ is finite dimensional.
Ex. Let \( X = \{ p : [0,1] \to \mathbb{R} \mid p \text{ is a polynomial} \} \).

Then \( (X, \| \cdot \|_\infty) \) is a n.I.S.

Define: \( T : X \to X \),

\[ \text{via } T(p) = p'. \]

\( T \) is a linear transformation.

Now let \( p_n = x^n + n \in \mathbb{N}. \)

Then \( \|p_n\|_\infty = 1. \) And

\[ \|T(p_n)\|_\infty = \|nx^{n-1}\|_\infty = n. \]

\[ \therefore T \notin \mathcal{L}(X). \]
Define the notion of a compact linear operator.

Spaces \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) are n.e.s.

A compact linear operator from \(X\) to \(Y\) is a linear transformation \(T\) from \(X\) to \(Y\) is compact in \((Y, \| \cdot \|_Y)\).

**Terminology:** "pre-compact" \(\equiv\) a set is pre-compact iff its closure is compact.

Then: \(T\) is a compact linear operator from \((X, \| \cdot \|_X)\) to \((Y, \| \cdot \|_Y)\) iff \(T\) is a linear transformation from \(X\) to \(Y\) and \(T(M)\) is compact in \((Y, \| \cdot \|_Y)\) whenever \(M\) is a bounded subset of \((X, \| \cdot \|_X)\).

**pf:** \((\Rightarrow)\) Immediate, since this implies \(T(B_1(0))\) is compact.

\((\Leftarrow)\) Suppose \(T\) is a compact linear operator from \((X, \| \cdot \|_X)\) to \((Y, \| \cdot \|_Y)\). Let \(M\) be bounded in \((X, \| \cdot \|_X)\). Then \(\exists R > 0\)

\[ M \subseteq B_R(0) \quad \therefore \quad T(M) \subseteq T(B_R(0)) \]
We will show \( T(B_r(0)) \)

is compact in \((Y, \| \cdot \|_Y)\). Let
\[
\{u_n\} \subseteq T(B_r(0)) = RT(B_r(0))
\]

We know \( T(B_r(0)) \) is compact in \((Y, \| \cdot \|_Y)\). Also, \( \{ru_n\} \subseteq T(B_r(0)) \)

\[
\Rightarrow \exists \{u_{n_k}\} \text{ a subseq of } \{u_n\}
\]

and \( \forall V \in T(B_r(0)) \Rightarrow \frac{1}{r} u_{n_k} \rightarrow V \)

\[
\Rightarrow u_{n_k} \rightarrow RV \in T(B_r(0))
\]

\( \therefore \) Done.

**The bottom line** is that a compact linear operator is a linear operator that sends bounded sets to pre-compact sets.

sets with compact closure
Notation: \( \text{Comp}(X,Y) \equiv \text{compact linear operators from } (X,\|\cdot\|_X) \to (Y,\|\cdot\|_Y) \).

If \( X = Y \) then we write \( \text{Comp}(X) \).

Show that compact linear operators are bounded, but bounded linear operators are not necessarily compact unless the space is finite dimensional.

**Theorem:** Space \((X,\|\cdot\|_X)\) and \((Y,\|\cdot\|_Y)\) are n.l.s. Then \( \text{Comp}(X,Y) \) is a subspace of \( \mathcal{L}(X,Y) \).

**Proof:** Let's show \( \text{Comp}(X,Y) \subseteq \mathcal{L}(X,Y) \). You show the subspace part.

Let \( T \in \text{Comp}(X,Y) \). Then

\[ T(B(0)) \text{ is compact.} \]

\[ \therefore T(B(0)) \text{ is bounded } \Rightarrow \exists K > 0 \ \forall \lambda \in Y \exists x \in X \ | \ |x|_X = 1 \]

\[ \|T(x)\|_Y \leq K \]

\[ \Rightarrow T \in \mathcal{L}(X,Y) \]

The converse is not true unless \( X \) is finite dimensional.

Ex. Let \((X,\|\cdot\|)\) be an infinite dim n.l.s. Then \( I : X \to X \) defined by \( I(x) = x \) is in \( \mathcal{L}(X) \).

But \( I(B(0)) = B(0) \) is not compact since \( X \) is infinite dim.
The Fredholm Alternative: A Very Powerful Result!!!

Then: Spec $(X, \| \cdot \|)$ is a n.l.s.

and $\text{Te} \text{Com}(X)$. If $f \in X$ then $\exists! \ x \in X$

iff $x - T(x) = f$

$\ker (I - T) = \{0\}$

Furthermore, when $\ker (I - T) = \{0\}$, $(I - T)^{-1}$ exists and $(I - T)^{-1} F(x)$

Note: $\rightarrow (I - T)x = f \iff x = (I - T)^{-1} f$

So, if $\ker (I - T) = \{0\}$, then the result above implies $f$

$K > 0$ independent of $f \in F$

$\|x\| \leq K \|f\|$. 
So, the Fredholm alternative says that if $T$ is a compact operator, then the nonhomogeneous problem has a unique solution if and only if the homogeneous problem only has the trivial solution.

\[ \text{i.e., } I - T \text{ is } 1-1, \text{ onto, } \quad \text{and } (I - T)^* \in L(X) \]

iff $I - T$ is 1-1.

Amazing, we get onto \( (I - T)^* \in L(X) \) for free!

The proof will be given next time (I hope)... ;)

Oct 14-7:48 AM
Application of the Fredholm Alternative:

Linear Initial Value Problems: Suppose \( A \in C(\mathbb{R}, \mathbb{R}^n) \) and \( f \in C(\mathbb{R}, \mathbb{R}^n) \) with \( u_0 \in \mathbb{R}^n \). Consider the problem of finding a function \( u \) so that

A system of 1st order linear differential equations

\[
\begin{align*}
\dot{u}(t) &= A(t)u(t) + f(t) \\
u(0) &= u_0,
\end{align*}
\]

Note: Our solution \( u \) should satisfy \( u: \mathbb{R} \to \mathbb{R}^n \) and \( u \in C([0, T], \mathbb{R}^n) \).

We will use the Fredholm alternative to prove this problem has a unique solution.

1. Rewrite the problem.
   \[
   \int_0^t u'(s) ds = \int_0^t A(s)u(s) ds + \int_0^t f(s) ds \]
   \[
   u(t) - u_0 = \int_0^t A(s)u(s) ds = \int_0^t f(s) ds
   \]

2. We show the original problem is equivalent to
   \[
   (P) \quad \begin{cases}
   \text{Find } u \in C([0, T], \mathbb{R}^n) \ni \\
   u(t) = \int_0^t A(s)u(s) ds = u_0 + \int_0^t f(s) ds
   \end{cases}
   \]

We will attack the problem by defining \( T(u) = \int_0^t A(s)u(s) ds \) for \( u \in C([a, b], \mathbb{R}^n) \equiv X \).

(here \( a, b \in \mathbb{R} \) with \( a < 0 < b \)).

We will see that \((X, \| \cdot \|_X)\) is a n.l.s. and \( T \in \text{Com}(X) \).
A key to showing $T$ is compact (as defined on the previous page) is the Ascoli-Arzela Theorem.

A very important consequence....

Recall $X = C^k([a,b], \mathbb{R}^n)$ with $k \in \mathbb{N}$ and

$$\| \cdot \| : X \to \mathbb{R}$$

given by

$$\| u \| = \sum_{j=0}^{k} \| u^{(j)} \|_{\infty}.$$

Bound subsets of $(X, \| \cdot \|)$ are pre-compact in

$$(C([a,b], \mathbb{R}^n), \| \cdot \|_{\infty}).$$
I will post some homework in the next few days...