Welcome Back!!

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The Fredholm Alternative: **A Very Powerful Result!!!**

Let \((X, \|\|)\) be a normed linear space and suppose \(T \in \text{Com}(X)\). Then either the homogeneous equation
\[
x - T(x) = 0 \quad \iff \quad (I - T)(x) = 0
\]
has a nontrivial solution, or for every \(y \in X\) the equation
\[
x - T(x) = y \quad \iff \quad (I - T)(x) = y
\]
has a unique solution and \((I - T)^{-1} \in L(X)\).

**Uniqueness \(\Rightarrow\) Existence**

So, the Fredholm alternative says that if \(T\) is a compact operator, then the nonhomogeneous problem has a unique solution if and only if the homogeneous problem only has the trivial solution.
Application of the Fredholm Alternative:

Linear Initial Value Problems: Suppose \( A \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n \times n}) \), \( f \in \mathcal{C}(\mathbb{R}, \mathbb{R}^n) \) and \( u_0 \in \mathbb{R}^n \). Consider the problem of finding a function \( u \) so that

\[
\begin{align*}
\dot{u}(t) &= A(t)u(t) + f(t) \\
u(0) &= u_0
\end{align*}
\]

We will use Fredholm alternative to prove this problem has a unique solution.

1. Re-frame and transform the problem. For \( T > 0 \) let

\[
\begin{align*}
\dot{u}(t) &= A(t)u(t) + f(t), \quad 0 \leq t \leq T \\
u(0) &= u_0
\end{align*}
\]

We will show \((\text{IVP})_T\) has a unique solution \( u \in \mathcal{C}([0, T], \mathbb{R}^n) \). This will guarantee the result for \((\text{IVP})\).

2. Rewrite \((\text{IVP})_T\). Integrate.

\[
\begin{align*}
0 &= u(t) - u(0) - \int_0^t A(s)u(s)\,ds - \int_0^t f(s)\,ds \\
u(t) - \int_0^t A(s)u(s)\,ds &= u_0 + \int_0^t f(s)\,ds
\end{align*}
\]

Define \( T(u)(t) \) and \( F(t) \),

\[
\begin{align*}
\begin{cases}
T(u)(t) = u(t) - \int_0^t A(s)u(s)\,ds \\
F(t) = \int_0^t f(s)\,ds
\end{cases}
\end{align*}
\]

Find \( u \in \mathcal{C}([0, T], \mathbb{R}^n) \) such that \( u - T(u) = F \).
\( (w) \quad \begin{cases} \text{Find } u \in C([-T, T], \mathbb{R}^n) \\ u - T(u) = F \end{cases} \)

1. \( T \) is a linear operator from \( C([-T, T], \mathbb{R}^n) \to C([-T, T], \mathbb{R}^n) \).

2. \( (w) \) is equiv. to \( (EVP)_T \).

3. Set \( X = C([-T, T], \mathbb{R}^n) \) and equip \( X \) with \( \| \cdot \|_{\infty} \). Then \((X, \| \cdot \|_{\infty})\) is a Banach Space, and we will show \( T \in \text{Com}(X) \).

4. Consider \( (WH) \begin{cases} \text{Find } u \in X \\ u - T(u) = 0 \end{cases} \)

We will show \( (WH) \) only has the trivial sol'n.

Note: \( \text{3) + 4) } \Rightarrow (w) \) has a unique sol'n via the Fredholm Alternative.
Recall: \( T: X \rightarrow X \)

Via

\[
T(u)(t) = \int_0^t a(s)u(s)ds.
\]

Q: How do we show \( T \) is compact?

A: We show that if \( Y \subseteq X \) is bounded, then \( T(Y) \) is precompact in \( X \).

We need to understand which subsets of \( X \) are precompact.
A key to showing $T$ is compact (as defined on the previous page) is the Ascoli-Arzela Theorem.

**Def.** Spec $a,b \in \mathbb{R}$ s.t. $a < b$. Set $Z = C([a,b], \mathbb{R}^n)$.

A set $U \subseteq Z$ is **equi-continuous** iff $\forall \varepsilon > 0 \exists \delta > 0$ s.t. $\forall x,y \in [a,b]$ with $|x-y| < \delta$, then $|u(x) - u(y)| < \varepsilon$ $\forall u \in U$.

**Theorem:** Let $Z = C([a,b], \mathbb{R}^n)$ equipped with $|| \cdot ||_{\infty}$. A subset $U \subseteq Z$ is pre-compact iff $U$ is bounded and equi-continuous. <idea of proof is coming.

**Cor.** Recall $C'([a,b], \mathbb{R}^n)$ and $|| \cdot ||_{\infty}$ defined by

$$||u|| = ||u||_{\infty} + ||u'||_{\infty}.$$

If $U \subseteq C'([a,b], \mathbb{R}^n)$ is ball wrt $|| \cdot ||_{\infty}$, then $U$ is precompact in $(Z, || \cdot ||_{\infty})$.

**why?** Note $U$ is bounded in $(Z, || \cdot ||_{\infty})$.

Is $U$ equi-continuous?

\[ \text{MVThm} \Rightarrow \text{if } x,y \in [a,b] \text{ then } \]

$$u(x) - u(y) = u'(c)(x-y)$$

for some $c$ between $x$ and $y$.

\[ \Rightarrow |u(x) - u(y)| \leq ||u'||_{\infty} |x-y| \]

Let $M > 0 \Rightarrow ||u|| \leq M \forall u \in U$. Then $|u(x) - u(y)| \leq M|x-y| \forall u \in U$.

\[ \therefore \text{ If } \varepsilon > 0, \text{ we can pick } s = \frac{\varepsilon}{M} \]

to see that $U$ is equi-continuous.
Let \( V \subseteq X \) be bounded w.r.t \( \| \cdot \|_\infty \).

Set \( U = T(V) \).

Claim: \( U \) is pre-compact.

Note: If \( v \in V \) then
\[
T(v)(t) = \int_0^t A(s)v(s)ds
\]

\[
\Rightarrow \left| T(v)(t) \right|_2 \leq |t| \max_{|s| \leq T} |A(s)v(s)|_2
\]
\[
= |t| \max_{|s| \leq T} \left| A(s)v(s) \right|_2
\]
\[
\leq M |t| \max_{|s| \leq T} |v(s)|_2
\]
\[
\therefore \quad \| T(V) \|_\infty \leq MT \| V \|_\infty
\]

i.e., \( T(V) \) is bounded w.r.t \( \| \cdot \|_\infty \).

Also, if \( v \in V \)
\[
\frac{d}{dt} T(v)(t) = A(t)v(t)
\]

\[\Rightarrow \left| \frac{d}{dt} T(v)(t) \right| \text{ is bounded}
\]

ind. of \( V \) since \( V \) is bounded w.r.t \( \| \cdot \|_\infty \).

\[\therefore \quad U = T(V) \text{ is bounded in } C'[\mathbb{R}] \text{ w.r.t } \| \cdot \|.
\]

So the cor \( \Rightarrow T \in \text{Conv}(X) \).
Theorem: Let $Z = C([a,b], \mathbb{R}^m)$ equipped with 1-norm. A subset $U \subseteq Z$ is pre-compact iff $U$ is bounded and equicontinuous. (idea of proof)

pf: ($\Rightarrow$) you do it.
($\Leftarrow$) Suppose $U$ is bounded and equicontinuous.
Let $\{V_k\} \subseteq U$. We need to show $\exists \{v_k\}$ and $\forall \varepsilon, z \exists v_k \to v$.
Let $\{a_\lambda\}_{\lambda=1}^\infty$ be dense in $[a,b]$. Then $\{V_k(a_\lambda)\} \subseteq \mathbb{R}^m$ is bounded, so it's a conv.
Subseq. $\{V_{k_{i_\lambda}}(a_{i_\lambda})\}$. Similarly, we can obtain $\{V_{k_{i_\lambda}}(a_{j_\lambda})\}$ as a conv.
Subseq. of $\{V_{k_{i_\lambda}}(a_{j_\lambda})\}$. Inductively, we obtain $\{V_{k_{i_\lambda+j_\lambda}}(a_{j_\lambda})\}$ as a conv.
Subseq. of $\{V_{k_{i_\lambda+j_\lambda}}(a_{j_\lambda})\}$.

Claim: $\{V_{k_{i_\lambda+j_\lambda}}\}$ is a Cauchy seq. in $Z$. Note: If we define $U_j = V_{k_{i_\lambda+j_\lambda}}$, then $\{U_j\}$ is bounded and equicontinuous, since it is contained in $U$. Also, $\{V_{k_{i_\lambda+j_\lambda}}\}$ is a subseq. of $\{V_k\}$.

Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. if $x, y \in [a,b]$ with $|x-y| < \delta$ then $|V_{k_{i_\lambda+j_\lambda}}(x) - V_{k_{i_\lambda+j_\lambda}}(y)| < \varepsilon / 3$.
Now let $L \subseteq \{a, \ldots, a_t\}$ is dense in $[a,b]$. i.e. $\forall x \in [a,b]$ $\exists \ell \in \{1, \ldots, t\}$ s.t. $|x - a_{\ell}| < \delta$. 

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Also, if \( i, j \geq N \) and \( l \in \{1, \ldots, L\} \) then
\[
|u_j(a_l) - u_i(a_l)| < \frac{\epsilon}{3}.
\]

Now let \( x \in [a, b] \). Choose \( l \in \{1, \ldots, L\} \) so that \( |x - a_l| < \delta \). Suppose \( i, j \geq N \). Then
\[
|u_i(x) - u_j(x)| \leq |u_i(x) - u_i(a_l)| + |u_i(a_l) - u_j(a_l)| + |u_j(a_l) - u_j(x)|
\]
\[
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

\( \therefore \) we get pointwise convergence of \( \{u_i(x)\} \) and \( x \in [a, b] \).

You can finish from here.
Let's prove #4:

Consider

$$\begin{cases}
\text{Find } u \in X, \\
u' + T(u) = 0
\end{cases}$$

We will show ($\forall u$) only has the trivial sol'n.

Note: $u - T(u) = 0$ if and only if

$$u(t) - \int_0^t A(s)u(s)\,ds = 0$$

for all $t \in [-T, T]$.

Q: How does this imply $u = 0$?

A: $u(t) = \int_0^t A(s)u(s)\,ds$

Suppose $t > 0$. I'll let you do

$$\frac{t}{2} \leq 0.$$ 

Then

$$|u(t)| \leq \int_0^t |A(s)u(s)|\,ds$$

$$\leq K \int_0^t |u(s)|\,ds$$

(since $A \in C([-T, T], \mathbb{R}^n)$)

$$0 \leq |u(t)| \leq K \int_0^t |u(s)|\,ds$$

$$\equiv w(t)$$

Note: $0 \leq w'(t) = K |u(t)| \leq kw(t)$

AND

$w(0) = 0$.
\[ w' - kw \leq 0 \]
\[
e^{-kt} w' - e^{-kt} kw \leq 0 \]

\[
\frac{d}{dt} (e^{-kt} w) \leq 0
\]

\[
e^{-kt} w(t) - e^{-kt} w(0) \leq 0
\]

\[
\Rightarrow w \equiv 0 \quad \text{since} \quad w \geq 0, \quad w(0) = 0.
\]

\[
\therefore \int_{0}^{t} |u(t)| \, dt = 0 \quad \forall t
\]

\[
\Rightarrow u \equiv 0.
\]