Welcome Back!!
The Fredholm Alternative: A Very Powerful Result!!!

Let $(X, \|\|)$ be a normed linear space and suppose $T \in \text{Com}(X)$. Then either the homogeneous equation

$$x - T(x) = 0 \iff (I - T)(x) = 0$$

has a nontrivial solution, or for every $y \in X$ the equation

$$x - T(x) = y \iff (I - T)^{-1}(x) = y$$

has a unique solution and $(I - T)^{-1} \in L(X)$.

So, $1-1 \Rightarrow$ onto and bad inverse.

So, the Fredholm alternative says that if $T$ is a compact operator, then the nonhomogeneous problem has a unique solution if and only if the homogeneous problem only has the trivial solution.

we will prove this today!
Last time, we used the Fredholm Alternative to prove the following problem has a unique solution:

**Linear Initial Value Problems:** Suppose \( A \in C(R, R^{n \times n}) \), \( f \in C(R, R^n) \) and \( u_0 \in R^n \). Consider the problem of finding a function \( u \) so that

\[
\begin{align*}
  u'(t) &= A(t)u(t) + f(t) \\
  u(0) &= u_0
\end{align*}
\]

\( u \in C([a, b], R^n) \) is the solution to the initial value problem (IVP).

A key to the proof was the Ascoli-Arzelà Theorem and its corollary.

Identifies the pre-compact subsets of \( C([a, b], R^n) \) with respect to \( \| \cdot \|_{\infty} \).

As the sets that are bounded and equicontinuous.

Let \( \| \cdot \| \) be the "natural norm" on \( C^1([a, b], R^n) \). i.e.

\[ \| u \| = \| u \|_{\infty} + \| u' \|_{\infty}. \]

Bounded subsets of \( (C^1([a, b], R^n), \| \cdot \|) \) are pre-compact in \( (C([a, b], R^n), \| \cdot \|_{\infty}) \).
Comments and Consequences: We know the following problem has a unique solution.

Linear Initial Value Problems: Suppose $A \in C(R, R^{n \times n})$, $f \in C(R, R^n)$ and $u_0 \in R^n$. Consider the problem of finding a function $u$ so that

$$
\begin{align*}
(IVP) \quad \left\{ 
& u'(t) = A(t)u(t) + f(t) \\
& u(0) = u_0
\end{align*}
$$

For $i \in \{1, \ldots, n\}$, $\vec{e}_i$ is the $i$th column of $I_{n \times n}$. Then $f! \quad u_i \in C^1(R, R^n) \Rightarrow$

$$
\begin{align*}
& u_i'(t) = A(t)u_i(t) \\
& u_i(0) = \vec{e}_i
\end{align*}
$$

Define $\mathbf{u} \in C^1(R, R^{n \times n}) \Rightarrow$

$$
\mathbf{u}(t) = \begin{pmatrix}
  u_1(t) \\
  u_2(t) \\
  \vdots \\
  u_n(t)
\end{pmatrix}
$$

Note:

$$
\mathbf{u}(0) = I_{n \times n}
$$

Also, $\mathbf{u}'(t) = \begin{pmatrix}
  u_1'(t) \\
  u_2'(t) \\
  \vdots \\
  u_n'(t)
\end{pmatrix}$

$$
= \begin{pmatrix}
  A(t)u_1(t) \\
  A(t)u_2(t) \\
  \vdots \\
  A(t)u_n(t)
\end{pmatrix}
$$

$$
= A(t)\mathbf{u}(t)
$$
\[ \begin{aligned}
\text{i.e.:} & \quad \begin{cases}
\Phi(t) = A(t) \Phi(0) \\
\Phi(0) = I_{n \times n}
\end{cases} \\
\text{Note: If} & \quad A(t) \equiv A, \quad \text{(ind. of t)} \\
\quad \text{then} & \quad \Phi(t) = e^{At}. \quad \text{(see Hwk 3)}
\end{aligned} \]

Back to (IVP).

(IVP) \quad \begin{cases}
u'(t) = A(t)u(t) + f(t) \\
u(0) = u_0
\end{cases}

Claim: Good guess.

Guess \neq 1

\[ u(t) = \Phi(t)u_0 + \int_0^t \Phi(t-s)f(s)ds \]

Try it:

\[ u'(t) = \Phi'(t)u_0 + \Phi(t-t)f(t) + \int_0^t \Phi'(t-s)f(s)ds \]

\[ = A(t)\Phi(t)u_0 + f(t) + \int_0^t A(t-s)\Phi(t-s)f(s)ds \]

Need

\[ = A(t)u(t) + f(t) \]

This doesn’t quite work.
Guess #2: Based on (IVP).

\[(\text{IVP}) \quad \begin{cases} u'(t) = A(t)u(t) + f(t) \\ u(0) = u_0 \end{cases}\]

\[u'(t) - A(t)u(t) = f(t)\]

\[\Phi(t)^{-1} \left[ u'(t) - A(t)u(t) \right] = \Phi(t)^{-1}f(t)\]

You can show \(\Phi(t)^{-1}\) exists.

\[\Phi(t)^{-1}u'(t) - \Phi(t)^{-1}A(t)u(t) = \Phi(t)^{-1}f(t)\]

?? \[\frac{d}{dt}(\Phi(t)^{-1}u(t))\]

If this works, then

\[\frac{d}{dt}(\Phi(t)^{-1}u(t)) = \Phi(t)^{-1}f(t)\]

\[\Phi(t)^{-1}u(t) - \Phi(t)^{-1}u(0) = \int_0^t \Phi(s)^{-1}f(s)ds\]

\[\implies u(t) = \Phi(t)^{-1}u(0) + \int_0^t \Phi(s)^{-1}f(s)ds\]

\[\implies u(t) = \Phi(t) \left[ u_0 + \int_0^t \Phi(s)^{-1}f(s)ds \right]\]

I'll let you check this in the next homework...
This time: The proof of the Fredholm Alternative.

Riesz Lemma: Let $X$ be a normed linear space and suppose $M$ is a closed proper subspace of $X$. If $0 \leq \theta < 1$ then there exists $x_\theta \in X$ such that $\|x_\theta\| = 1$ and $\text{dist}(x_\theta, M) \geq \theta$.

Remark: Suppose for the moment that $X$ is an i.p.s.

![Diagram]

$z \in M^\perp \Rightarrow \|z\| = 1$.

Then $\text{dist}(z, M) = 1$.

The Riesz lemma says that a normed linear space almost has this property. It just falls a little short.

Proof: Let $x \in X \setminus M$. Since $M$ is closed $\text{dist}(x, M) > 0$. Set $d = \text{dist}(x, M)$. Note: The result is easy if $\theta = 0$. Use any unit vector for $x_\theta$. Suppose $0 < \theta < 1$.

Then $\frac{d}{\theta} > d = \text{dist}(x, M)$. \[ \theta \]

If $y_\theta \in M \Rightarrow \|x - y_\theta\| \leq \frac{d}{\theta}.$

Now create $x_\theta = \frac{1}{\|x - y_\theta\|}(x - y_\theta)$. $\|x_\theta\| = 1$. 

Oct 28-5:47 AM
Also, for \( y \in M \) we have

\[
\| x_0 - y \| = \left\| \frac{1}{\| x - y_0 \|} (x - y_0) - y \right\|
\]

\[
= \frac{1}{\| x - y_0 \|} \left\| x - y_0 - \left( \frac{1}{\| x - y_0 \|} \right) (x - y_0) \right\|
\]

\[
\geq \frac{d}{\| x - y_0 \|} \geq \theta.
\]

\[\#\]

 Pf. of F. A. : 3 parts. We will prove:

1. If \( N = \{ x \in X \mid x - T(x) = 0 \} \) then

   \[
   \exists \ k > 0 \Rightarrow \text{dist}(x, N) \leq k \| x - T(x) \| \ \forall \ x \in X.
   \]

   \[\text{Note: If } N \text{ is trivial } \| x \| \leq k \| (I - T) x \| \]

   So if \((I - T)^{-1}\) exists then

   \[
   \| (I - T)^{-1} y \| \leq k \| y \|
   \]

   \[
   \Rightarrow (I - T)^{-1} y \in \mathcal{L}(X).
   \]

2. \( R(I - T) \) is closed.

3. If \( N = \{ 0 \} \) then \( R(I - T) = X \).
Proof of ii: Suppose not. Then \( \exists \{x_n\}_{n=1}^{\infty} \in X \)
\[ \lim_{n \to \infty} \|x_n - T(x_n)\| = 0 \quad \text{and} \quad \exists d_n \in \text{dist}(x_n, N) \Rightarrow d_n \Rightarrow 0 \]
\[ \text{Choose \( \{y_n\}_{n=1}^{\infty} \in N \) such that} \quad d_n = 1 \quad \text{and} \quad \|x_n - y_n\| \leq 2d_n \]

Set \( z_n = \frac{1}{\|y_n - y_n\|} (x_n - y_n) \). Then \( \|z_n\| = 1 \) and \( \|z_n - Tz_n\| = \frac{1}{\|x_n - y_n\|} \|x_n - y_n\| - 2 \)
\[ \leq \frac{1}{d_n} \to 0 \]

\[ \therefore \quad z_n - T(z_n) \to 0 \]

Also, \( T \)

is compact and \( \{z_n\} \) is bad.

\[ \exists \{z_{n_k}\} \quad \text{and} \quad \exists y_0 \in X \]

\[ T(z_{n_k}) \to y_0 \]

\[ \therefore \quad z_{n_k} - T(z_{n_k}) \to 0 \]

\[ \quad \downarrow \]

\[ y_0 \]

\[ \Rightarrow \quad z_{n_k} \to y_0 \]

\[ \therefore \quad y_0 - T(y_0) = 0 \]

\[ \Rightarrow \quad y_0 \in N. \quad \text{This creates trouble} \]

b/c

\[ \text{dist}(z_{n_k}, N) = \inf_{y \in N} \|z_{n_k} - y\| \]

\[ = \frac{1}{\|x_{n_k} - y_{n_k}\|} \inf_{y \in N} \|x_{n_k} - y_{n_k} - (x_{n_k} - y_{n_k}) - y\| \]

\[ = \frac{1}{\|x_{n_k} - y_{n_k}\|} \text{dist}(x_{n_k}, N) \geq \frac{1}{2} d_{n_k} \]

Contradiction!!
Recall \( \bar{u} \). \( A(I-T) \) is closed.

Let \( y \in A(I-T) \). Let \( \{x_n\} \in X \ni x_n - T(x_n) \to y \). Use (i).

\( \text{dist}(x_n, N) \) is bad ind. of \( n \).

Let \( d_n = \text{dist}(x_n, N) \). Choose \( \exists \{y_n\} \subseteq N \ni d_n \leq ||x_n - y_n|| \leq 2d_n \).

Set \( w_n = x_n - y_n \). Then \( \{w_n\} \) is bad (b/c \( d_n \)'s unif bad)

Also

\( w_n - T(w_n) = x_n - T(x_n) \to y \).

\( T \) is compact. \( \exists \{w_{n_k}\} \) and \( w_0 \in X \ni T(w_{n_k}) \to w_0 \). Then

\[ w_{n_k} - T(w_{n_k}) \to y \]

\[ \downarrow \]

\[ w_0 \]

\[ \Rightarrow w_{n_k} \to w_0 + y \]

\[ \therefore (y + w_0) - T(y + w_0) = y \]

\[ \Rightarrow y \in A(I-T) \Rightarrow A(I-T) \]

is closed. \( \Box \) is done.
If \( N = \{0\} \), then \( R/(I-T) = X \).

Set \( R_j = (I-T)^j X \) for \( j \in N \).

\( \Rightarrow \) \( R_j \) is a closed subspace of \( X \times j \). Also \( R_j \supseteq R_{j+1} \times j \) since \( R_{j+1} \supseteq R_j \)

\( j \in N \). From a lemma \( \exists \{y_n\} \subseteq X \)

\( \exists y_n \in R_n \), \( \|y_n\| = 1 \), \( \text{dist}(y_n, R_{n+1}) > \frac{1}{2} \).

\( \Rightarrow \) If \( n > m \) then \( \exists R_n \subseteq R_{m+1} \)

\( T(y_m) - T(y_n) = y_n + (y_n - (y_m - T(y_m))) + (y_m - T(y_m)) \)

= \( y_m - y \) for some \( y \in R_{m+1} \)

\( \Rightarrow \|T(y_m) - T(y_n)\| = \|y_m - y\| \)

\( \geq \text{dist}(y_m, R_{m+1}) \)

= \( \frac{1}{2} \).

This causes a problem with the compactness of \( T \).
\[ \exists k \in \mathbb{N} \geq R_j = R_k \ \forall \ j \geq k. \]

Let \( y \in X \). Then \((I-T)^k(y) = R_k = R_{k+1}\)

So \( \exists x \in X \) \( (I-T)^k(y) = (I-T)^{k+1}(x) \)

i.e.

\[ (I-T)^k(y - (I-T)(x)) = 0. \]

\[ y - (I-T)(x) = 0 \]

i.e.

\[ x - T(x) = y. \]

So \( y \in R(I-T). \)