Welcome Back...

Questions?
4. Let \( k \in \mathbb{N} \) and suppose \( X \subseteq H^k((0,1), R) \) is closed, nonempty and convex in \( X \) whenever \( \alpha \geq 0 \) and \( u \in X \). Let \( l : X \to [0, \infty) \) be continuous (with respect to the natural norm on \( H^k \)) and satisfy \( l(\alpha u) = \alpha l(u) \) for all \( \alpha \geq 0 \) and \( u \in X \). Prove that if \( l(p) \neq 0 \) for all \( p \in X \) such that \( p \) is a nonzero polynomial of degree at most \( k - 1 \), then there exists \( L > 0 \) such that
\[
\|u\|_{2,0,(0,1)} \leq L \left[ \|u^{(k)}\|_{2,(0,1)} + l(u) \right]
\]
for all \( u \in X \). Explain how this result generalizes the Poincaré inequality.

\( X \) is a cone.

Note that a subspace is a cone, but a cone need not be a subspace.

\( \text{e.g.} \)

- Take \( k=1 \). Set \( X = H^1((0,1), \mathbb{R}) \).

\( \text{Let } l(u) = \left| \int_0^1 u(x) \, dx \right| \).

\( \text{Note: } l(1) = 1. \quad \text{Also, } l(\alpha u) = \alpha l(u) \quad \forall \alpha \neq 0. \)

\( \therefore \exists L > 0 \quad \text{such that} \)
\[
\|u\|_{2,1,(0,1)} \leq L \left[ \left( \int_0^1 (u'(x))^2 \, dx \right)^{1/2} + l(u) \right]
\]
\( \rightarrow \)

\( H^1 \text{ norm of } u \)
\[ \kappa = 1 \quad \text{or} \quad X = \left\{ u \in H^1((0,1), \mathbb{R}) \mid \int_0^1 u(x) \, dx = 0 \right\} \]

Take \( \lambda(u) \equiv 0 \).

Note: There are no nonzero constant functions in \( X \).

\[ \therefore \lambda(p) \neq 0 \text{ for a nonzero constant} \]

functions in \( X \).

So, \( \exists L > 0 \) such that

\[ \|u\|_{L^2((0,1))} \leq L \left( \int_0^1 (u'(x))^2 \, dx \right)^{1/2} \]

\[ \therefore \text{this is a norm on } X \]

equivalent to \( \|u\|_{L^2((0,1))} \).
\[ X = \{ u \in H'(0,1), \mathbb{R} \mid u(0) = u(1) = 0 \} \]

\[ H_0^1(0,1), \mathbb{R} \]

Define \( \lambda(u) = 0 \) \( \forall \ u \in X \).

Note: There are no nonzero constant functions in \( X \).

\( \Rightarrow \lambda(p) \neq 0 \) \( \forall \) nonzero constant functions in \( X \).

\[ \exists L > 0 \quad \|u\|_{2,1,(0,1)} \leq L \left( \int_0^1 u'(x)^2 \, dx \right)^{1/2} \]

i.e., \( \|u\|_{2,1,(0,1)} \) is a norm on \( H_0^1(0,1), \mathbb{R} \) that is equivalent to the standard \( H' \) norm.

Poincare ineq.
The proof can be done by contradiction.

Spec \( L > 0 \) and 
\[
\|u\|_{2,K,(0,1)} \leq L \left( \|u^{(k)}\|_{2,(0,1)} + \|v\| \right)
\]

\( u \in X \). Then \( \forall \, n \in \mathbb{N} \)

\( \exists \, u_n \in X \)

\[
\|u_n\|_{2,K,(0,1)} > \sqrt{\|u^{(k)}\|_{2,(0,1)} + \alpha \|u_n\|_{2,(0,1)}}
\]

Define 
\[
\alpha = \frac{1}{\|u_n\|_{2,K,(0,1)}}
\]

\[
\alpha \|u_n\|_{2,K,(0,1)} > \sqrt{\|u^{(k)}\|_{2,(0,1)} + \alpha \|u_n\|_{2,(0,1)}}
\]

\[
L = \|v\|_{2,K,(0,1)} > \sqrt{\|V^{(k)}\|_{2,(0,1)} + \alpha \|v\|_{2,(0,1)}}
\]

\[\text{Case } k=1: \{V_n\} \text{ is a bad seq. in } H^1((0,1), \mathbb{R}).\]

Then \( \exists \) a subseq. \( \{V_{n_j}\} \) that converges in \( C([0,1],\mathbb{R}) \) wrt \( \|\cdot\|_{0,\infty} \).

Let \( v \in C([0,1],\mathbb{R}) \) s.t. \( V_{n_j} \rightarrow v \) wrt \( \|\cdot\|_{0,\infty} \).

Also, \( \|V_{n_j}\|_{2,(0,1)} \rightarrow 0 \) as \( j \rightarrow \infty \).

Use this to prove \( v \) is a constant and (from \( \circ \) and the continuity of \( \lambda \) ) \( \lambda(v) = 0 \).

And \( v \) is nonzero since \( \|v\|_{2,(0,1)} = 1 \).

Contradiction! So \( k=1 \) is true.
Define $X = \{ u \in H^1((0,1), R) \mid \int_0^1 u(x) dx = 0 \}$. Let $f \in L^2((0,1), R)$ and $p \in C^1([0,1], R)$ such that $p > 0$. Give a necessary and sufficient condition on $f$ to guarantee there exists a unique function $u \in H^2((0,1), R) \cap X$ such that

$$
\begin{align*}
\frac{d}{dx} (p(x) u'(x)) &= f(x), \quad 0 < x < 1 \\
u'(0) = u'(1) &= 0
\end{align*}
$$

Then show there exists a real number $K > 0$ dependent upon $p$, but independent of $f$, such that

$$
\|u\|_{2,2,(0,1)} \leq K \|f\|_{2,(0,1)}.
$$

Finally, show that if we define a sequence of functions $\{u_n\}_{n=1}^{\infty} \subseteq H^2((0,1), R) \cap X$ such that

$$
\begin{align*}
\frac{d}{dx} (p(x) u'_n(x)) - \frac{1}{x} u_n(x) &= f(x), \quad 0 < x < 1 \\
u'_n(0) = u'_n(1) &= 0
\end{align*}
$$

for each $n \in N$, then $u_n \rightarrow u$ as $n \rightarrow \infty$.

Get a weak formulation on $X$. Show $f \in X$ solving.

Continue...
Today: We will prove the Spectral Theorem for Compact Operators

Recall Riesz' Lemma:
Let $X$ be a n.l.s. and space $M \subseteq X$ is a proper closed subspace. If $0 < \theta < 1$ then $\exists x_\theta \in X$ such that $\|x_\theta\| = 1$ and $\text{dist}(x_\theta, M) \geq \theta$.

Theorem (Riesz-Schauder):
Space $X$ is a nls over $F$ and $T \in \text{COM}(X)$. Then

(i) $0 \in \sigma(T)$ if $X$ is infinite dim.

(ii) if $\lambda \in F \setminus \{0\}$ then either $\lambda$ is an eigenvalue or $\lambda \in \rho(T)$.

(iii) $T$ has at most countably many eigenvalues, and if there are infinitely many eigenvalues, $0$ is the only limit point.

(iv) if $\lambda \in \sigma(T)$ is nonzero (i.e., $\lambda$ is an eigenvalue from (ii)) then $\text{dim}(E_\lambda) < \infty$.

i.e., $X$ has finite multiplicity.

Note that any uncountable set of real or complex numbers must have a nonzero accumulation point.
pf: As I said... (i) is done.

Let's show (ii). Assume \( \dim(X) = \infty \).

We will show \( \emptyset \in \sigma(T) \).

By way of contradiction, spec \( \emptyset \in \rho(T) \).

Let's build a sequence. Take \( x_1 \in X \ni \|x_1\| = 1 \). Set \( M_1 = \text{span}\{x_1\} \).

\[ \therefore \text{by Riesz lemma } \exists \ x_2 \in X \ni \|x_2\| = 1 \text{ and } \text{dist}(x_2, M_1) > \frac{1}{2} . \]

Take \( M_2 = \text{span}\{x_1, x_2\} \) and \( \exists \ x_3 \in X \ni \|x_3\| = 1 \text{ and } \text{dist}(x_3, M_2) > \frac{1}{2} . \)

Continue to create \( \{x_n\} \subseteq X \ni \|x_n\| = 1 \text{ and } \text{dist}(x_n, \text{span}\{x_1, \ldots, x_{n-1}\}) > \frac{1}{2} . \)

Then \( \{x_n\} \) is an infinite sequence with no conv. subseq.

Define \( y_n = T^{-1}(x_n) \) for \( n \in \mathbb{N} \).

Note that since \( \emptyset \in \rho(T) \) we have \( \{y_n\} \) is a bdd seq. in \( X \).

Recall: \( T \in \text{COM}(X) \). :\[
\{T(y_n)\} \quad \text{must have a conv. subseq. i.e. } \{x_n\} \text{ must have a conv. subseq.}
\]

\[ \therefore (i) \text{ is true.} \]
let's do (iii) + (iv) together.

Spe \{x_n\} \subseteq X is a seq of
L.I. eig. vectors with assoc. Nonzero
eig. vals \{\lambda_n\} \Rightarrow \lambda_n \to \lambda.

Lemma: Any uncountable set of real or complex numbers must have a
nonzero accumulation point.

If \{S\} be the set. Define

1. \( S_1 = \{x \in S \mid |x| < 1\} \).
2. \( S_2 = \{x \in S \mid 1 < |x| < \frac{1}{2}\} \).

For \( k > 2 \)

\( S_k = \{x \in S \mid \frac{1}{k+1} < |x| < \frac{1}{k}\} \)

Note: \( S = \emptyset \cup S_1 \cup S_2 \cup \ldots \)

countable union.

\* at least one \( S_k \) is
uncountable. If \( k > 1 \)

Then \( S_k \) is \( \text{bld} + \text{infinite} \)
and \( \text{dist}(S_k, 0) \geq \frac{1}{k} \).

\( \Rightarrow \) \( S_k \) has a nonzero
accum. pt. \( \Rightarrow \) \( S \) has also.

Suppose \( k = 1 \). Then \( S_1 \) is
uncountable. Define

\( A_k = \{x \in S_1 \mid k \leq |x| < k+1\} \).

Then \( S_1 = U A_k \)

\( \Rightarrow \) at least one \( A_k \)’s
is uncountable.

Note: Each \( A_k \) is bld
and \( \text{dist}(A_k, 0) \geq 1 \).

Done. #
Claim: $\lambda = 0$.

Set $M_k = \text{span} \{x_1, x_2, \ldots, x_k\}$. Then

$\lambda_n \neq M_n$, e $\|y_n\| = 1$ and

$$\text{dist}(y_n, M_{n-1}) > \frac{1}{2}.$$ 

If $n > m$

$$\lambda_n^{-1}T(y_n) - \lambda_m^{-1}T(y_m) = y_n + (-y_m - \lambda_m^{-1}(\lambda_m I - T)(y_m)) + \lambda_m^{-1}(\lambda_m I - T)y_m$$

$$= y_n - z \text{ for some } z \in M_{n-1}$$

$$\therefore \|\lambda_n^{-1}T(y_n) - \lambda_m^{-1}T(y_m)\| > \frac{1}{2}.$$ 

Recall $\lambda_n \rightarrow \lambda$. 

Now suppose $\lambda \neq 0$. Recall, $T \in \text{Com}(X)$ and \{\tilde{y}_k\} is add.

Then $\{T(y_k)\}$ has a conv. subseq. Contradiction to $\bigstar$.

$\therefore \lambda = 0. \Rightarrow (iii) + (iv)$ hold.
Final remark: If $\lambda \in \mathbb{F}$ and $x \in X$ s.t. $\|x\|=1$ and $T(x) = \lambda x$

$\Rightarrow \|T(x)\| = |\lambda| \|x\| = |\lambda|$

$\Rightarrow |\lambda| \leq \|T\|_0.$