Math 1431

$f'(x) = r(x)$

$g'(x) = s(x)$

Test 4 – Practice Problems

1. Suppose $f(x)$ is an anti-derivative of $r(x)$, and $g(x)$ is an anti-derivative of $s(x)$. We are given the data in the table about the functions f, g, r and s.

x	1	2	3	4
$f(x)$	3	2	1	4
$r(x)$	1	4	2	3
$g(x)$	2	1	4	3
$s(x)$	4	2	3	1

$f(3) = 1$

$g(3) = 4$

$f(1) = 3$

$g(1) = 2$

a) $\int_1^3 (3r(x) - 2s(x))\,dx = \int_1^3 (3f'(x) - 2g'(x))\,dx$

$= (3f(x) - 2g(x))\Big|_1^3$

$= (3\cdot f(3) - 2\cdot g(3)) - (3f(1) - 2g(1))$

$= (3 - 8) - (9 - 4) = -10.$

$g(4) = 3$

$g(1) = 2$

b) Give the average value of $s(x)$ on the interval $[1,4]$.

$= \frac{1}{4-1}\int_1^4 s(x)\,dx = \frac{1}{3} g(x)\Big|_1^4$

recall: $g'(x) = s(x)$

$= \frac{1}{3}(g(4) - g(1)) = \frac{1}{3}(3-2)$

$= \frac{1}{3}.$

Average value of $s(x)$ on $[1,4]$.

2. Suppose $F''(x) = x^2 - \frac{2}{\sqrt{x}} + 1$, $F'(1) = -3$ and $F(1) = 2$. Find $F(x)$.

$\frac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$

$F'(x) = \frac{1}{3}x^3 - 2 \cdot 2x^{1/2} + x + C_1 \Rightarrow -3 = \frac{1}{3} - 4 + 1 + C_1$

$\Rightarrow C_1 = -\frac{1}{3} \quad \Rightarrow F'(x) = \frac{1}{3}x^3 - 4x^{1/2} + x - \frac{1}{3}$

$\Rightarrow F(x) = \frac{1}{12}x^4 - 4 \cdot \frac{2}{3}x^{3/2} + \frac{1}{2}x^2 - \frac{1}{3}x + C_2$, $\quad 2 = \frac{1}{12} - \frac{8}{3} + \frac{1}{2} - \frac{1}{3} + C_2$

Find C_2.
Rewrite $F(x)$.

3. Evaluate: $\frac{d}{dx}\int_{-1}^{2x^3} \sin(t^2)\,dt$ and $\frac{d}{dx}\int_{1-2x}^{2x^3} \sin(t^2)\,dt$.

$= \sin((2x^3)^2) \cdot 6x^2 = 6x^2 \sin(4x^6)$

$= \frac{d}{dx}\int_{1-2x}^{500} \sin(t^2)\,dt + \frac{d}{dx}\int_{500}^{2x^3} \sin(t^2)\,dt$

The value 500 could be ANY number.

$= -\frac{d}{dx}\int_{500}^{1-2x} \sin(t^2)\,dt + 6x^2\sin(4x^6)$

$= -\sin((1-2x)^2) \cdot (-2) + 6x^2 \sin(4x^6) = 2\sin((1-2x)^2) + 6x^2\sin(4x^6)$

4. Find a formula for $f(x)$: $2x^3 - 3x^2 + x - 1 = \int_{-1}^{x} f(t)\,dt$

Differentiate.

$\boxed{6x^2 - 6x + 1 = f(x)}$

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\,u'(x)$$

Note:

$$\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt$$

5. $\int_2^7 x\sqrt{x^2+2}\,dx = \frac{1}{2}\int_2^7 \underbrace{(x^2+2)}_{u}^{1/2}\underbrace{2x\,dx}_{du}$

$u = x^2+2$

$du = 2x\,dx$

$x=7 \Rightarrow u=51$

$x=2 \Rightarrow u=6$

$$= \frac{1}{2}\int_{\boxed{6}}^{\boxed{51}} u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_6^{51}$$

$$= \frac{1}{3}\left(51^{3/2} - 6^{3/2}\right).$$

6. $\int \left(3\sec^2(2x) - 2\sqrt{x-1}\right)dx = \int 3\sec^2(2x)\,dx - \int 2\sqrt{x-1}\,dx$

you can use u-sub if you aren't this comfortable.

$$= \frac{3}{2}\tan(2x) - 2\cdot\frac{2}{3}(x-1)^{3/2} + C$$

$$= \frac{3}{2}\tan(2x) - \frac{4}{3}(x-1)^{3/2} + C$$

7. Give the average value of $f(x) = x^2 - 2x + 4$ on the interval [-1,2], and verify the conclusion of the mean value theorem for integrals for this function on this interval.

$$= \frac{1}{2-{-1}}\int_{-1}^{2}(x^2-2x+4)\,dx = \frac{1}{3}\left(\frac{1}{3}x^3 - x^2 + 4x\right)\Big|_{-1}^{2}$$

$$= \frac{1}{3}\left[\left(\frac{8}{3} - 4 + 8\right) - \left(-\frac{1}{3} - 1 - 4\right)\right]$$

$$= \frac{1}{3}\left[3 + 9\right] = 4$$

$\leftarrow$ Average value of $f(x)$ on $[-1,2]$.

Now, find a value c so that $-1 < c < 2$ and $f(c) = 4$.

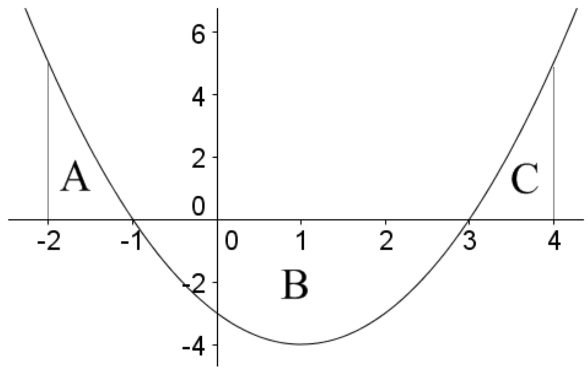
Solve $c^2 - 2c + 4 = 4 \Rightarrow c^2 - 2c = 0$

$c(c-2) = 0$

$c = 0$ or $\cancel{c = 2}$

Note: $-1 < 0 < 2$.

8. The graph of $f(x)$ is shown below. The area of region A is 7/3, the area of region B is 34/3, and the area of region C is 7/3.



a. Give the area of the region bounded between the graph of $f(x)$ and the x-axis on the interval [-2 , 4].

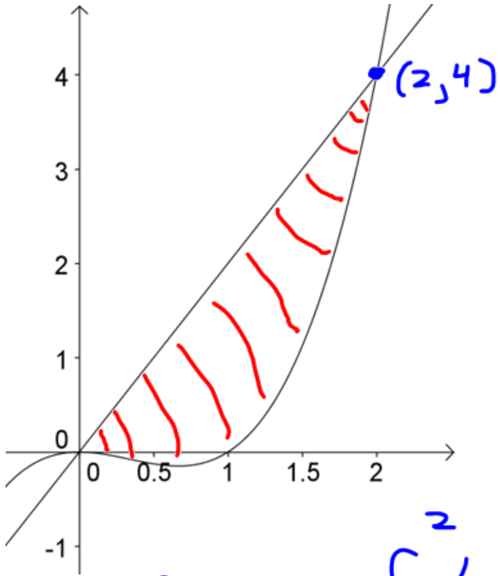
Area(A) + Area(B) + Area(C)

$= \frac{7}{3} + \frac{34}{3} + \frac{7}{3} = \frac{48}{3}$

b. $\int_{-1}^{4} f(x)dx =$ − Area(B) + Area(C)

$= -\frac{34}{3} + \frac{7}{3} = -9.$

9. Find the **area** bounded between the graphs of $f(x) = x^3 - x^2$ and $g(x) = 2x$ on the interval [0,2]. The graph is shown below.



$x^3 - x^2 = 2x$

$x^3 - x^2 - 2x = 0$

$x(x^2 - x - 2) = 0$

$x(x-2)(x+1) = 0$

$$\text{Area} = \int_0^2 (\text{Top} - \text{Bottom})\,dx$$

$$= \int_0^2 (2x - (x^3 - x^2))\,dx$$

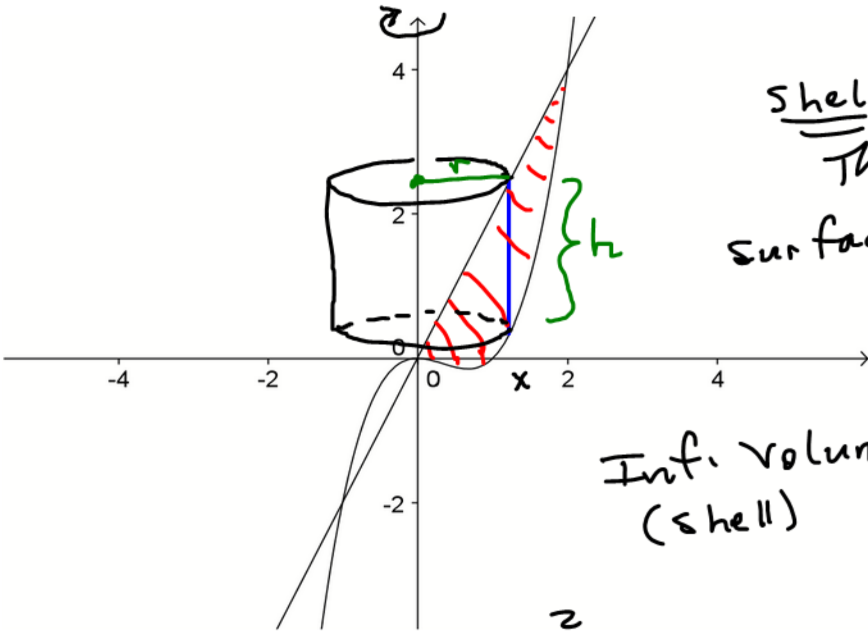
$$= \int_0^2 (2x - x^3 + x^2)\,dx = \left(x^2 - \tfrac{1}{4}x^4 + \tfrac{1}{3}x^3\right)\Big|_0^2$$

$$= 4 - 4 + \tfrac{8}{3} - 0$$

$$= \tfrac{8}{3}.$$

$y = x^3 - x^2$ $\quad$ $y = 2x$

10. The region bounded between the graphs of $f(x) = x^3 - x^2$ and $g(x) = 2x$ on the interval [0,2] (see the previous problem) is rotated around the y-axis to generate a solid. Find the volume.



Shell

Thickness $= dx$

Surface Area $= 2\pi r h$

$= 2\pi x\left(2x - (x^3 - x^2)\right)$

Infi. Volume (shell) $= 2\pi x\,(2x - x^3 + x^2)\,dx$

$$\text{Full Volume} = \int_0^2 2\pi x\,(2x - x^3 + x^2)\,dx$$

$$= 2\pi \int_0^2 (2x^2 - x^4 + x^3)\,dx$$

$$= 2\pi\left(\frac{2}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{4}x^4\right)\Big|_0^2$$

$$= 2\pi\left[\left(\frac{16}{3} - \frac{32}{5} + 4\right) - 0\right] = \text{you do it.}$$

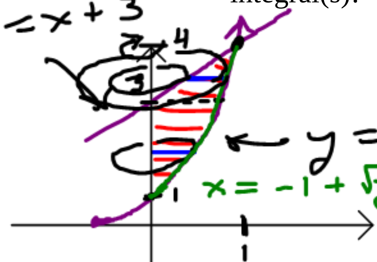
11. Sketch the region in the first quadrant bounded between the graphs of $f(x) = x+3$ and $g(x) = (x+1)^2$. Then rotate this region around the y-axis to generate a solid.

$x = y - 3$

$y = x + 3$

a. Give a formula involving integral(s) in y for the volume generated. Do not compute the integral(s)!

→ dy integration (Need horiz. line segs)



$y = (x+1)^2$

$x = -1 + \sqrt{y}$

$\pm\sqrt{y} = x+1$

$-1 \pm \sqrt{y} = x$

$x+3 = (x+1)^2$

$x+3 = x^2+2x+1$

$x^2+x-2=0$

$(x+2)(x-1)=0$

$x=-2,\ x=1$

$1 \le y \le 3$ Disk,

Thickness $= dy$

Area $= \pi r^2$

$= \pi(-1+\sqrt{y})^2$

Inf. volume $= \pi(-1+\sqrt{y})^2 dy$

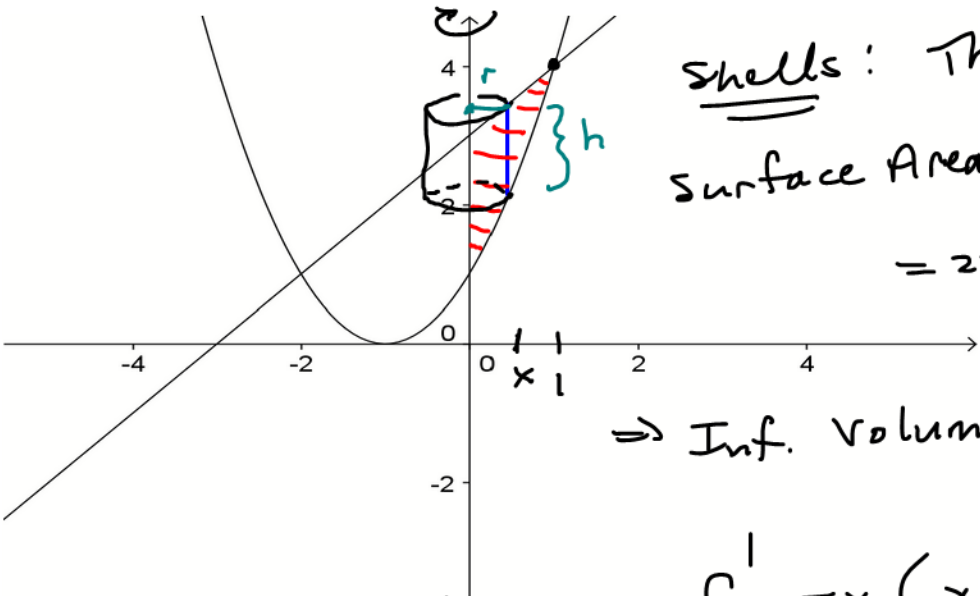
$3 \le y \le 4$ washer

See next page.

b. Give a formula involving integral(s) in x for the volume generated. Do not compute the integral(s)!

→ dx integration

use vertical line segs.



Shells: Thickness $= dx$

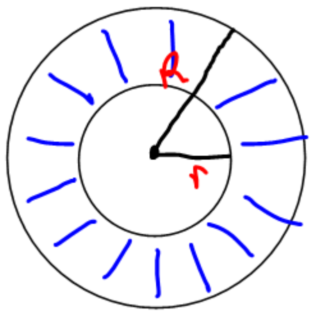
Surface Area $= 2\pi r h$, $r = x$, $h = (x+3-(x+1)^2)$

$= 2\pi x\,(x+3-(x+1)^2)$

$\Rightarrow$ Inf. Volume $= 2\pi x\,(x+3-(x+1)^2)\,dx$

$$\text{Full Volume} = \int_0^1 2\pi x\,(x+3-(x+1)^2)\,dx$$

Washer $(3 \le y \le 4)$



Thickness = dy

Area $= \pi(\ R\ )^2 - \pi(\ r\ )^2$

$= \pi(-1+\sqrt{y})^2 - \pi(y-3)^2$

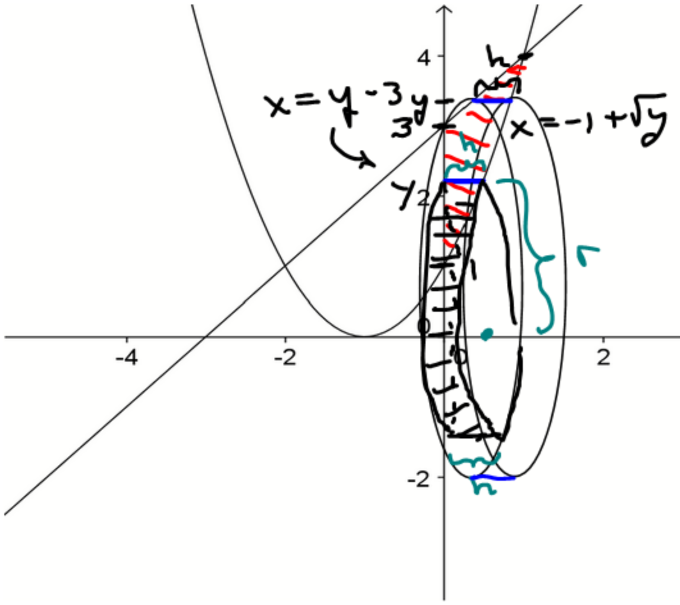
Inf. Volume $= \left(\pi(-1+\sqrt{y})^2 - \pi(y-3)^2\right) dy$

$r = y-3$

$R = -1+\sqrt{y}$

$$\text{Full volume} = \underbrace{\int_1^3 \pi(-1+\sqrt{y})^2\, dy}_{1 \le y \le 3} + \underbrace{\int_3^4 \left(\pi(-1+\sqrt{y})^2 - \pi(y-3)^2\right) dy}_{3 \le y \le 4}$$

12. Repeat the previous problem, assuming that the region is rotated around the *x*-axis to generate a volume.



a) dy integral(s)

$\Rightarrow$ horiz line segs.

$1 \le y \le 3$

shell.

Thickness $= dy$

Surface area $= 2\pi r h$

$= 2\pi y(-1+\sqrt{y})$

Inf. Volume $= 2\pi y(-1+\sqrt{y})\,dy$

$3 \le y \le 4$ shell

Thickness $= dy$

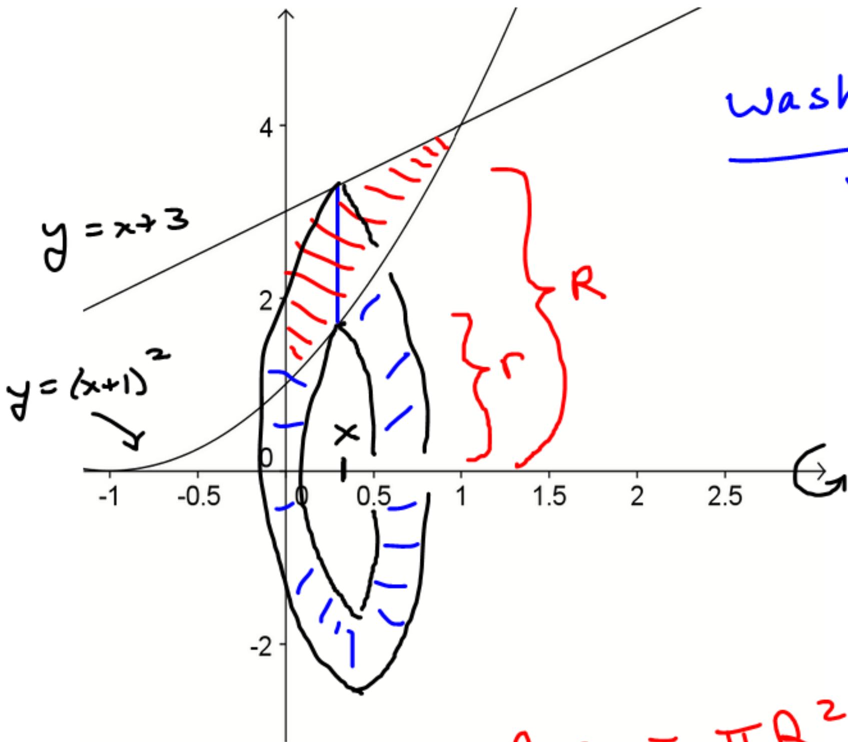
Surface Area $= 2\pi r h$

$= 2\pi y((-1+\sqrt{y}) - (y-3))$

Inf. Volume $= 2\pi y(2+\sqrt{y}-y)\,dy$

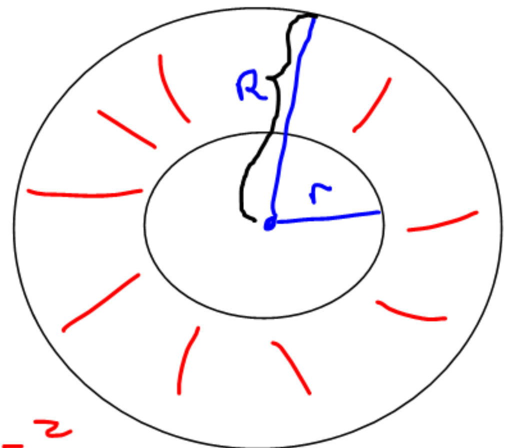
$$\text{Full Volume} = \underbrace{\int_1^3 2\pi y(-1+\sqrt{y})\,dy}_{1 \le y \le 3} + \underbrace{\int_3^4 2\pi y(2+\sqrt{y}-y)\,dy}_{3 \le y \le 4}$$

b) dx integration. $\Rightarrow$ vertical line segments



washers:

Thickness $= dx$



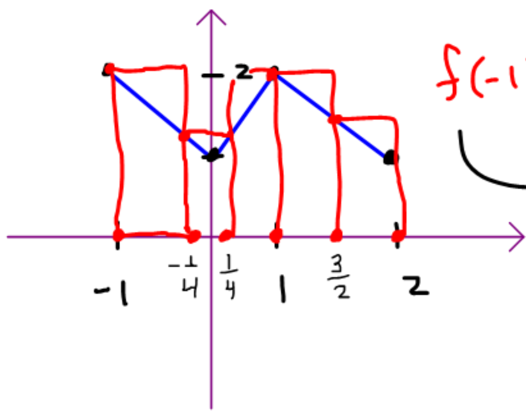
$$\text{Area} = \pi R^2 - \pi r^2$$

$$= \pi (x+3)^2 - \pi \left((x+1)^2\right)^2$$

$$\text{Inf. volume (washer)} = \left(\pi (x+3)^2 - \pi (x+1)^4\right) dx$$

$$\text{Full volume} = \int_0^1 \left(\pi (x+3)^2 - \pi (x+1)^4\right) dx \; .$$

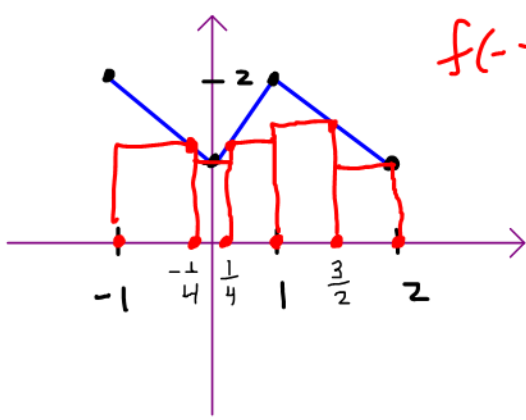
13. a. Give the Upper Riemann sum for the function $f(x)=\begin{cases}1-x, & -1\le x<0\\ x+1, & 0\le x<1\\ 3-x, & 1\le x\le 2\end{cases}$ on the interval [-1,2] with respect to the partition P={-1, -1/4, 1/4, 1, 3/2, 2}.



$$\underbrace{f(-1)\cdot\frac{3}{4}+f(-\tfrac{1}{4})\cdot\frac{1}{2}+f(1)\cdot\frac{3}{4}+f(1)\frac{1}{2}+f(\tfrac{3}{2})\cdot\frac{1}{2}}_{\text{put the values in.}}$$

$\vdots$

b. Give the Lower Riemann sum for the function $f(x)=\begin{cases}1-x, & -1\le x<0\\ x+1, & 0\le x<1\\ 3-x, & 1\le x\le 2\end{cases}$ on the interval [-1,2] with respect to the partition P={-1, -1/4, 1/4, 1, 3/2, 2}.



$$f(-\tfrac{1}{4})\cdot\frac{3}{4}+f(0)\cdot\frac{1}{2}+f(\tfrac{1}{4})\cdot\frac{3}{4}+f(\tfrac{3}{2})\cdot\frac{1}{2}+f(2)\cdot\frac{1}{2}$$

Get the values

$\vdots$

14. Sketch the region bounded between the graphs of $x + y = 2$ and $x = y^2$. Then give a formula for the area of the region involving integral(s) in x. Repeat the process with integral(s) in y.