Yet another....
Test 2 Review

Material: Everything from the beginning of the semester, through section 3.8.

Note: There will not be any problems on falling bodies, extreme value theorem or $\varepsilon\delta$ proofs.

Test Structure: The Test appears to have 10 questions, with both 9 and 10 being related rate questions. However, you will be able to choose whether you solve question 9 or 10. Consequently, there are only 9 questions. There are 3 multiple choice questions and 6 written questions. You will receive partial credit on the written questions. A few of the questions have more than one part.

$\frac{0}{0}$

$$\lim_{x\to 0} \frac{5x}{\sin(2x)} = \lim_{x\to 0} \frac{5}{2} \cdot \frac{2x}{\sin(2x)} = \frac{5}{2}$$

$\frac{2x}{\sin(2x)} \to 1$

$\frac{0}{0}$

$$\lim_{x\to 1} \underbrace{\frac{x^2-1}{x^2-4x+3}}_{\text{rational function}} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-3)} = \frac{2}{-2} = -1$$

$$\lim_{x\to 0} \frac{\sin(4x)}{\sin(2x)} = \lim_{x\to 0} \frac{4}{2} \cdot \frac{2x}{\sin(2x)} \frac{\sin(4x)}{4x} = 2$$

$\frac{2x}{\sin(2x)} \to 1$, $\frac{\sin(4x)}{4x} \to 1$

$\frac{-1}{0}$

$$\lim_{x\to 2} \underbrace{\frac{x-3}{x^2+3x-10}}_{\text{rational function}} = \text{dne}$$

b/c the numerator $\to$ nonzero and denominator $\to 0$.

$\frac{\sin\left(\frac{3\pi}{2}\right)}{2\pi}$

$$\lim_{x\to \pi/2} \frac{\sin(3x)}{4x} = -\frac{1}{2\pi}$$

$$\boxed{\lim_{u\to 0} \frac{\sin(u)}{u} = 1}$$

$\frac{0}{1}$

$$\lim_{x\to 0} \frac{2x}{\cos(3x)} = 0$$

$\frac{0}{0}$

$$\lim_{x\to 0} \frac{(1-\cos(x))}{x^2} \frac{(1+\cos(x))}{(1+\cos(x))} = \lim_{x\to 0} \frac{1-\cos^2(x)}{x^2(1+\cos(x))}$$

$$= \lim_{x\to 0} \frac{\sin^2(x)}{x^2(1+\cos(x))} = \lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)^2 \cdot \frac{1}{1+\cos(x)}$$

$\frac{\sin(x)}{x} \to 1$, $\frac{1}{1+\cos(x)} \to \frac{1}{2}$

$$= \frac{1}{2}$$

Give values of A and B so that the function

$$f(x) = \begin{cases} Ax + 1, & x < 0 \\ -1, & x = 0 \\ Ax^2 - 2B, & x > 0 \end{cases}$$

is continuous

We only need to check $x=0$.

Need:

$$f(0) = \lim_{x \to 0} f(x)$$

$$\lim_{x \to 0^-} (Ax+1) = 1$$

$$\lim_{x \to 0^+} (Ax^2 - 2B) = -2B$$

-1

Wow! $-1 \neq 1$.

$\therefore$ Regardless of A and B, f will never be cont. at $x=0$.

Note: If $B = -\frac{1}{2}$, then the discont at $x=0$ is removable.

If $B \neq -\frac{1}{2}$, then the discont is a jump at $x=0$.

Give the set on which the function $f(x) = \underbrace{\frac{x^2 - 3x - 4}{x^2 - 16}}$ is continuous.

rational function

$\therefore$ $f(x)$ is continuous everywhere it is defined.

i.e. everywhere except where

$$x^2 - 16 = 0$$

$$(x-4)(x+4) = 0$$

$$x = 4, \; x = -4.$$

f is continuous on

$$(-\infty, -4) \cup (-4, 4) \cup (4, \infty).$$

$$f(x) = \frac{x^2 - 3x - 4}{x^2 - 16}$$

Question: What types of discont do we have at $x=-4$ and $x=4$.

Note: $f(x) = \dfrac{x^2-3x-4}{x^2-16}$

$= \dfrac{\cancel{(x-4)}(x+1)}{\cancel{(x-4)}(x+4)}, \quad x \neq 4$

Removable discont at $x=4$.

Infinite discont at $x=-4$.

Use the intermediate value theorem to show there is a solution to the equation $\underbrace{x^4 - 3x + 1}_{} = \underline{\underline{7}}$ on the interval [-2,1].

$f(x) = x^4 - 3x + 1 \leftarrow$ polynomial.

$\updownarrow$

continuous everywhere.

<u>Check</u>: $f(-2) = 23$

$f(1) = -1$

<u>note</u>: 7 is btwn 23 and -1.

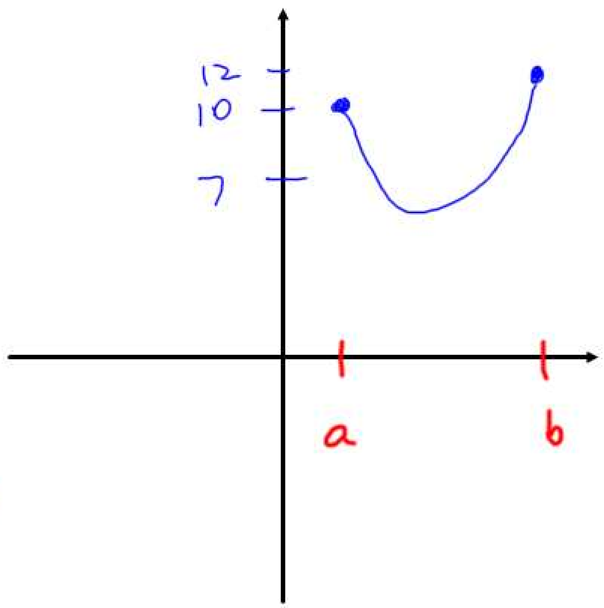
$\therefore$ by the IVThm there is a value c btwn -2 and 1 where $f(c) = 7$.

$\nwarrow$ our sol'n.

Aside:

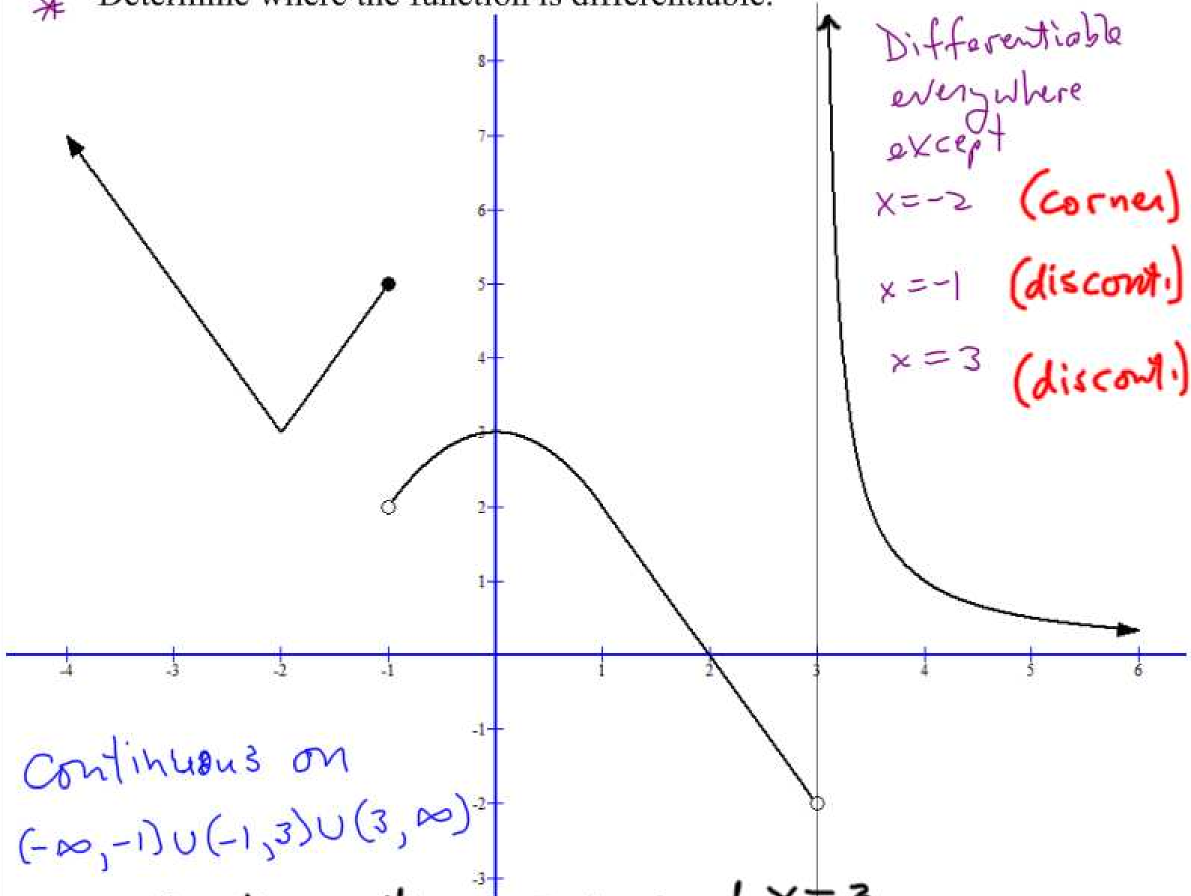
$f(x) = 7$

Not the function on the previous page.



Determine where the function is continuous, and classify any discontinuities. Explain your answers.

* Determine where the function is differentiable.



Differentiable everywhere except

$x=-2$ (corner)

$x=-1$ (discont.)

$x=3$ (discont.)

Continuous on

$(-\infty,-1)\cup(-1,3)\cup(3,\infty)$

Discontinuities at $x=-1$ and $x=3$

$\underline{x=-1}$: $\boxed{\text{Jump}}$ discontinuity.

b/c $\lim\limits_{x\to -1^-} f(x) = 5$ ← Not equal.

and $\lim\limits_{x\to -1^+} f(x) = 2$

$\underline{x=3}$: Infinite discontinuity.

b/c there is a vertical asymptote from the right.

$$\frac{d}{dx}\underbrace{(\sin(2x)\cos(3x))}_{\text{product}} = \sin(2x)\left(-3\sin(3x)\right) + \cos(3x)\,2\cos(2x)$$

$$= -3\sin(2x)\sin(3x) + 2\cos(3x)\cos(2x)$$

$$\frac{d}{dx}\left(\sin(x - 3\sqrt{x}) + \cos(2x^2 + 1)\right) =$$

$$= \cos(x - 3\sqrt{x}) \cdot \left(1 - \frac{3}{2\sqrt{x}}\right) - \sin(2x^2+1) \cdot 4x$$

$$= \left(1 - \frac{3}{2\sqrt{x}}\right)\cos(x - 3\sqrt{x}) - 4x\sin(2x^2+1)$$

$$\frac{d}{dx}\left(2\sin(3x) + \tan(2x)\right)^5 =$$

$$5\left(2\sin(3x) + \tan(2x)\right)^4\left(6\cos(3x) + 2\sec^2(2x)\right)$$

$$\frac{d}{dx}\left(\cos(2x) + x\right)^4 = 4\left(\cos(2x) + x\right)^3 \cdot \left(-2\sin(2x) + 1\right)$$

$y = \cos(2x) - \sqrt{x}$

Find $\frac{dy}{dx}$

$$\equiv \frac{d}{dx}\left(\cos(2x) - \sqrt{x}\right)$$

Find $\frac{d^2}{dx^2}\left(2x^2 - 3\sqrt{x} + \sin(2x)\right)$

Find $\frac{d}{dx} f(g(x))$ at $x = 2$, where $g(2) = 3$, $f(2) = -1$, $g'(2) = 6$ and $f'(3) = -4$.

1. $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

2. $\left.\frac{d}{dx} f(g(x))\right|_{x=2} = f'(g(2))\,g'(2)$

$= f'(3) \cdot 6$

$= -4 \cdot (6) = -24$

Note:

Find $\frac{d}{dx} f(g(h(x)))$

$$= f'(\underbrace{g(h(x))}) \, g'(\underbrace{h(x)}) \, h'(x)$$

Let $G(x)=\dfrac{f(x)}{g(x)}+3\left(f(x)+x\right)^2$. Suppose $f(1)=2$, $g(1)=-3$, $f'(1)=-2$ and $g'(1)=3$. Give $G'(1)$.

1. $G'(x)=\dfrac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}+6\left(f(x)+x\right)\left(f'(x)+1\right)$

2. $G'(1)=\dfrac{g(1)f'(1)-f(1)g'(1)}{g(1)^2}+6\left(f(1)+1\right)\left(f'(1)+1\right)$

plug in

$=\dfrac{(-3)(-2)-(2)(3)}{9}+6\left(2+1\right)\left(-2+1\right)$

$=\ldots$

Give a formula for $\frac{dy}{dx}$ in terms of x and y, given that $x^3 - 3xy + 2y^3 = 6$.

Treat y like a diff function of x.

$$\frac{d}{dx}\left(x^3 - \underline{3xy} + \underline{2y^3}\right) = \frac{d}{dx}6$$

$$\underset{=}{3x^2} - \left[3x\underset{=}{\frac{dy}{dx}} + \underset{=}{y\cdot 3}\right] + 6y^2\underset{=}{\frac{dy}{dx}} = 0$$

$$\frac{dy}{dx}\left(-3x + 6y^2\right) = -3x^2 + 3y$$

$$\frac{dy}{dx} = \frac{-3x^2 + 3y}{-3x + 6y^2}$$

$$f(x) = \frac{a}{bx+c} \quad \text{or} \quad f(x) = ax^2+bx+c$$
$$\text{or} \quad f(x) = \sqrt{ax+b}$$

Use the definition of the derivative to find the derivative of $f(x) = \sqrt{x-1}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h}$$

"$\frac{0}{0}$" ind. form

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h-1} - \sqrt{x-1}\right)\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{(x+h-1) - (x-1)}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-1} + \sqrt{x-1}\right)} = \frac{1}{2\sqrt{x-1}}$$

Use the definition of derivative to find the derivative of

$$f(x) = -2x^2 + 3x - 1.$$

$$f'(x) = -4x+3$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-2(x+h)^2 + 3(x+h) - 1 - (-2x^2+3x-1)}{h}$$

" $\frac{0}{0}$ " ind. form

$$= \lim_{h \to 0} \frac{-2(x^2+2xh+h^2) + 3x + 3h - 1 + 2x^2 - 3x + 1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-2x^2} - 4xh - 2h^2 \cancel{+3x} + 3h \cancel{-1} \cancel{+2x^2} \cancel{-3x} \cancel{+1}}{h}$$

$$= \lim_{h \to 0} \frac{-4x\cancel{h} - 2h^{\cancel{2}} + 3\cancel{h}}{\cancel{h}}$$

$$= \lim_{h \to 0} (-4x - 2h + 3) = -4x + 3$$

Find the equation of the tangent line to the graph of

$$x^2 + \cos(x + y) + y = \pi - 1$$

at the point $(0,\pi)$.

1. Find $\frac{dy}{dx}$ at $(0, \pi)$.

$$\frac{d}{dx}\left(x^2 + \cos(x+y) + y\right) = \frac{d}{dx}(\pi - 1)$$

$$2x + \left(-\sin(x+y)\cdot\left(1 + \frac{dy}{dx}\right)\right) + \frac{dy}{dx} = 0$$

Subst $x = 0$, $y = \pi$

$$0 + 0 + \frac{dy}{dx} = 0$$

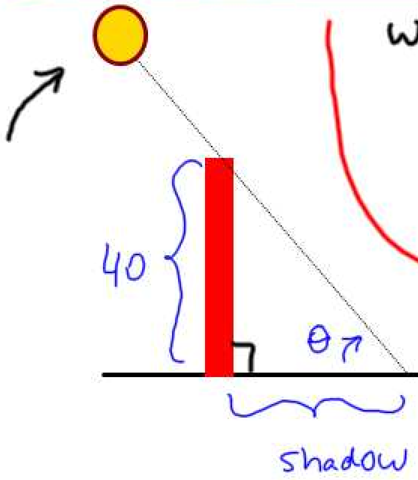
$$\Rightarrow \left.\frac{dy}{dx}\right|_{(0,\pi)} = 0$$ ← T.L. is horizontal

Tangent line at $(0, \pi)$:

$$y - \pi = 0(x - 0)$$ ←

$$\boxed{y = \pi}$$

On a morning of a day when the sun will pass directly overhead, the shadow of an 40-ft building on level ground is 30 feet long. At the moment in question, the angle the sun makes with the ground is increasing at the rate of $\pi/1500$ radians/minute. At what rate is the shadow length decreasing?



when $x = 30$ ft,

$$\frac{d\theta}{dt} = \frac{\pi}{1500} \text{ rad/min.}$$

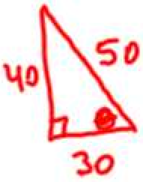
Find $-\frac{dx}{dt}$.

$$\tan(\theta) = \frac{40}{x}$$

$$\cot(\theta) = \frac{x}{40}$$

Diff wrt t.

$$\underbrace{-\csc^2(\theta)}_{??}\,\underbrace{\frac{d\theta}{dt}}_{\frac{\pi}{1500}} = \frac{1}{40}\,\underbrace{\frac{dx}{dt}}_{\text{Find } -\frac{dx}{dt} \text{ when } x=30}$$

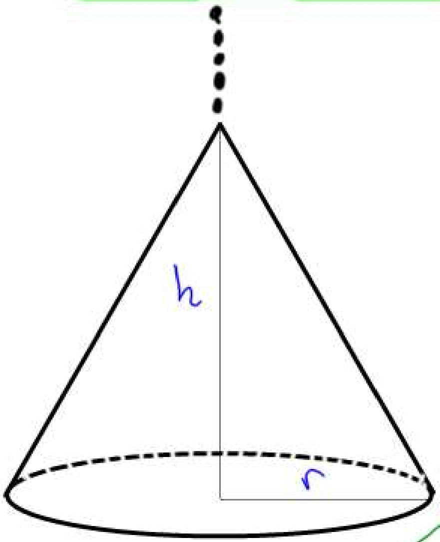


$$\csc(\theta) = \frac{50}{40} = \frac{5}{4}$$

$$-\frac{25}{16} \cdot \frac{\pi}{1500} = \frac{1}{40}\frac{dx}{dt}$$

$$\Rightarrow \boxed{-\frac{dx}{dt} = \frac{2\pi}{48} = \frac{\pi}{24} \text{ ft/min}}$$

Gravel is being poured by a conveyor onto a conical pile at the constant rate of 60π cubic feet per minute. Frictional forces within the pile are such that the height of the pile is always 1/3 of the radius. How fast is the radius of the pile changing at the instant when the radius is 5 feet? How fast is the height of the pile changing at the instant when the radius is 5 feet?



$V \equiv$ Volume of Sand at time t

$$\frac{dV}{dt} = 60\pi \text{ ft}^3/\text{min}$$

$$\boxed{h = \frac{1}{3} r}$$

Find $\frac{dr}{dt}$ when $r = 5$ ft.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{9}\pi r^3$$

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dr}{dt}$$

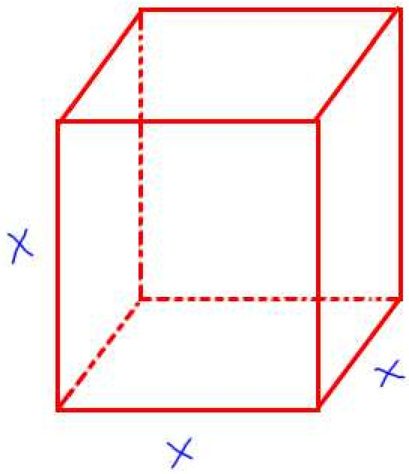
60π 5   Find

$$60\pi = \frac{1}{3}\pi \cdot 25 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{180}{25}$$

$$\Rightarrow \frac{dr}{dt} = \frac{36}{5} \text{ ft/min}$$

$$\frac{dh}{dt} = \frac{1}{3}\frac{dr}{dt} = \frac{12}{5} \text{ ft/min.}$$

A pile of trash in the shape of a cube is being compacted into a smaller cube. Suppose the volume is always a cube, and the volume is decreasing at the rate of 2 cubic meters per minute. Find the rate of change of the surface area of the cube at the instant when the volume is 27 cubic meters.



$V = x^3$, $S = 6x^2$

$\frac{dV}{dt} = -2 \ m^3/\text{min.}$

Find $\frac{dS}{dt}$ when $V = 27 \, m^3$

$V = 27 \Rightarrow x = 3$

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

-2   3   ☺

$-2 = 27 \frac{dx}{dt}$

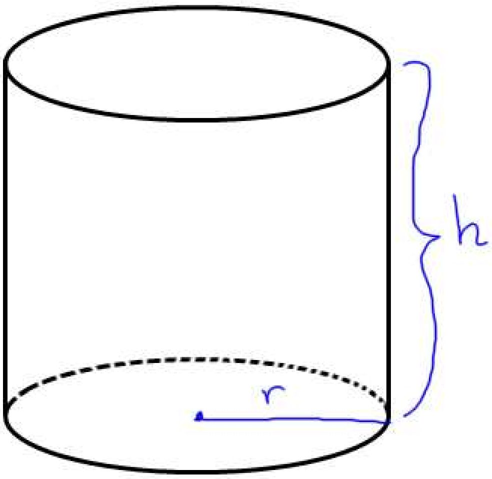
$\frac{dx}{dt} = -\frac{2}{27}$

$\frac{dS}{dt} = 12x \frac{dx}{dt}$

Find   ? 3   ? $\frac{-2}{27}$

$\Rightarrow \frac{dS}{dt} = 36 \frac{(-2)}{27} = \frac{-8}{3} \ m^2/\text{min}$

The diameter and height of a right circular cylinder are found at a certain instant to be 10 cm and 20 cm, respectively. If the diameter is increasing at the rate of 3 cm/sec, what change in height will keep the volume constant?



If $\frac{d}{dt}2r = 3$ cm/sec

then find $\frac{dh}{dt}$

so that $\frac{dV}{dt} = 0$.

Note: $2r = 10$ cm

and $h = 20$ cm

at a certain instant.

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + h \cdot 2\pi r \frac{dr}{dt}$$

$\frac{dV}{dt} = 0$, $r = 5$, $\frac{dh}{dt}$: Find, $h = 20$, $r = 5$, $\frac{dr}{dt} = \frac{3}{2}$

plug + solve.