Math 1431

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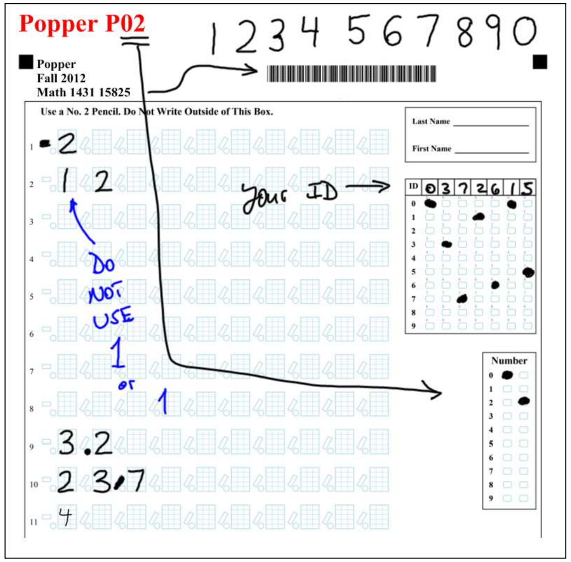
Online Quizzes are available.

Homework 3 is due next Monday in lab/workshop.

EMCF07 is due Friday morning at 9:00am.

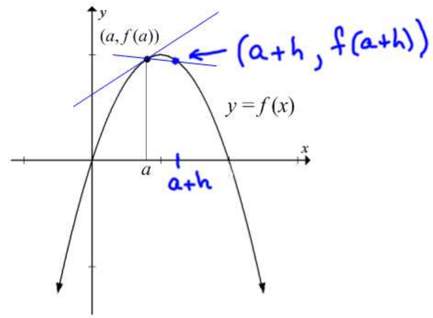
There is a **Written Quiz** in lab/workshop on Friday.

Popper P02



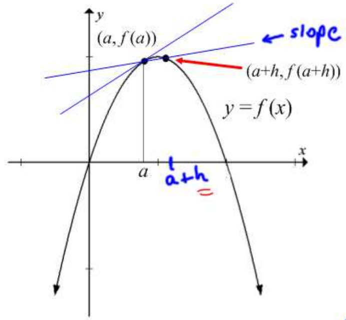
An Introduction to Derivatives: How can we approximate the slope of the tangent line to the graph at $x = a$?

I.



We can approximate the tangent line to the graph at $x = a$ by using a secant line.

II.

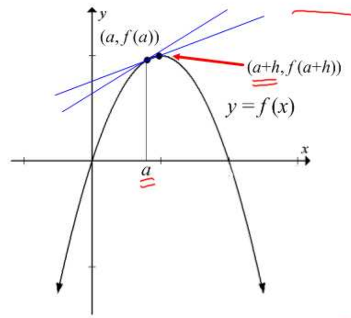


$\leftarrow$ slope $= \dfrac{f(a+h)-f(a)}{a+h-a}$

$$\text{slope of the secant line} = \frac{f(a+h)-f(a)}{h} = \frac{\Delta y}{\Delta x}$$

We can improve this approximation by making h smaller.

III.

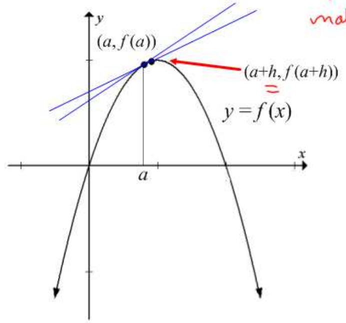


$$\text{slope of the secant line} = \frac{f(a+h)-f(a)}{h}$$

The approximation will continue to improve as we make h even smaller.

i.e. make h closer to 0.

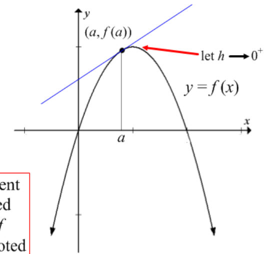
IV.



$$\text{slope of the secant line} = \frac{f(a+h)-f(a)}{h}$$

If the limit exists, then we can find the slope of the graph at $x = a$ by taking a limit.

V.



The slope of the tangent line at $x = a$ is called **the derivative** of f at $x = a$, and it is denoted by $f\,'(a)$.

$$\text{slope of the tangent line at at } x=a = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$= f'(a)$

"f prime at a"

Example: Find the derivative of $f(x) = x^2$ at $x = 1$, and give the equation of the tangent line to the graph of f at the point where $x = 1$.

① Find $f'(1)$ ← slope of the T.L. at $x=1$.

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^2 - 1}{h}$$

" $\frac{0}{0}$ " ind. form

$$= \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{2\cancel{h} + h^{\cancel{2}}}{\cancel{h}} = \lim_{h \to 0} (2+h) = 2$$

$\therefore \quad f'(1) = 2$

i.e. the slope of the tangent line at $x=1$ is 2.

$f(x) = x^2$

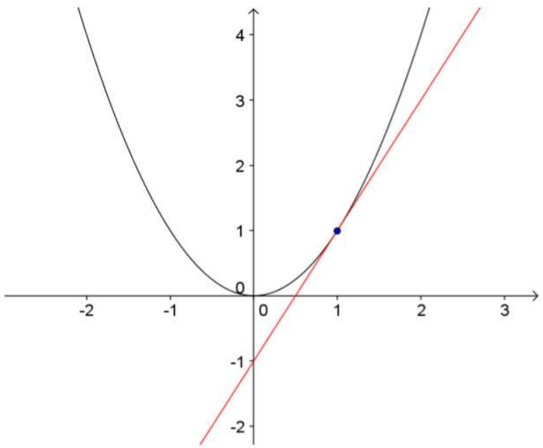
② slope $= f'(1) = 2$

Point $= (1, f(1)) = (1, 1)$

Equation: $\boxed{y - 1 = 2(x - 1)}$

$y = 2x - 1$

The graph of $f(x)$ and its tangent line at $x = 1$.



Example: Find the derivative of $f(x) = x^2$.

↑ formula

↳ $f'(x) = 2x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

↑ slope function

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

" $\frac{0}{0}$ ind form"

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}} = \lim_{h \to 0} (2x+h) = 2x$$

The Derivative: Overview...

$$\underbrace{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{\text{definition of derivative}}$$

$\dfrac{df(x)}{dx}$

Other notation: $\dfrac{d}{dx} f(x) = f'(x)$

Note: $f'(a)$ is the slope of the tangent line at $x = a$.

Interpretations of the Derivative

Function	Description	Derivative	Interpretation
$F(x)$	Standard function.	$F'(x)$	Slope function.
$F(x)$	Standard function.	$F'(x)$	Instantaneous rate of change of $F(x)$ with respect to x.
$s(t)$	Position at time t.	$s'(t)$	Velocity, sometimes named $v(t)$.
$v(t)$	Velocity at time t.	$v'(t)$	Acceleration, sometimes named $a(t)$.

Notation: $f(x)$ is differentiable at $x = a$ if and only if $f'(a)$ exists.
(i.e. if and only if f has a derivative at $x = a$)

Example: Give a formula for the derivative of $f(x) = \dfrac{1}{x+1}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$$

"$\frac{0}{0}$" indi form

$$= \lim_{h \to 0} \frac{\frac{x+1-(x+h+1)}{(x+h+1)(x+1)}}{h}$$

$\dfrac{\frac{a}{b}}{c} = \dfrac{a}{bc}$

$$= \lim_{h \to 0} \frac{\cancel{x}+\cancel{1}-\cancel{x}-h-\cancel{1}}{h(x+h+1)(x+1)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(x+h+1)(x+1)}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h+1)(x+1)} = \frac{-1}{(x+1)^2}$$

$\therefore$ for $f(x) = \dfrac{1}{x+1}$, we have $f'(x) = \dfrac{-1}{(x+1)^2}$

Example: Give an equation for the tangent line to the graph of $f(x) = \dfrac{1}{x+1}$ at $x = 2$.

Recall: $f'(x) = \dfrac{-1}{(x+1)^2}$

Slope $= f'(2) = -\dfrac{1}{9}$

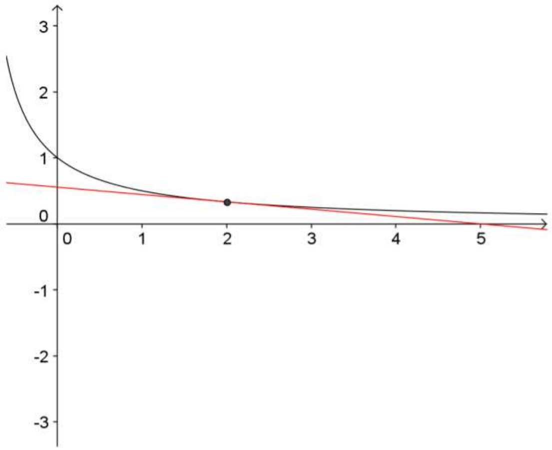
Point $= (2, f(2)) = (2, \frac{1}{3})$

Equation of T.L.: $y - \frac{1}{3} = -\frac{1}{9}(x-2)$

$y - \frac{1}{3} = -\frac{1}{9}x + \frac{2}{9}$

$y = -\frac{1}{9}x + \frac{5}{9}$

The graph of $f(x)$ and its tangent line at $x = 2$.



How can we use the derivative to find the slope of the normal line to the graph of $f(x)$ at $x = a$?

The normal line is perpendicular to the tangent line, and it passes through $(a, f(a))$.

Normal Line Slope $= \dfrac{-1}{f'(a)}$

provided $f'(a) \neq 0$.

Example: Give an equation for the normal line to the graph of $f(x) = \dfrac{1}{x+1}$ at $x = 1$.

Recall: $f'(x) = \dfrac{-1}{(x+1)^2} \Rightarrow f'(1) = -\frac{1}{4}$

slope: $\dfrac{-1}{f'(1)} = 4$

Point: $(1, f(1)) = (1, \frac{1}{2})$

$\boxed{\text{Equation for N.L.:}}$ $y - \frac{1}{2} = 4(x-1)$

$y = 4x - 4 + \frac{1}{2}$

$\boxed{y = 4x - \frac{7}{2}}$

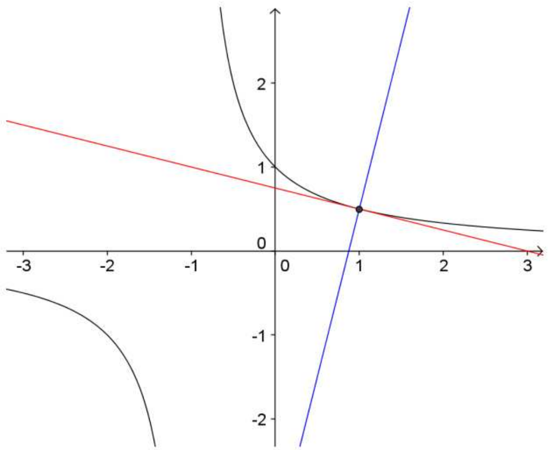
Note — Eq for T.L.: slope $= -\frac{1}{4}$

$y - \frac{1}{2} = -\frac{1}{4}(x-1)$

$y = -\frac{1}{4}x + \frac{1}{4} + \frac{1}{2}$

$\boxed{y = -\frac{1}{4}x + \frac{3}{4}}$

The graph of $f(x)$, along with its normal and tangent lines at $x = 1$.



How is the derivative related to continuity?

If $f'(a)$ exists then f is continuous at $x=a$.

Recall:
$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

So, we must have

$$\lim_{h\to 0}\left(f(a+h)-f(a)\right)=0$$

(otherwise this does not exist)

i.e. $\lim_{h\to 0} f(a+h) = f(a)$

Differentiability $\Rightarrow$ Continuity.

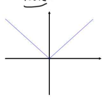
The converse is not true.

ex. $f(x)=|x|$ at $x=0$.

"corner" at $x=0$.

f is cont. at $x=0$, but "slope" makes no sense at $x=0$.

Note: $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$



$$= \lim_{h\to 0} \frac{|h|-0}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h} = \text{dne}$$

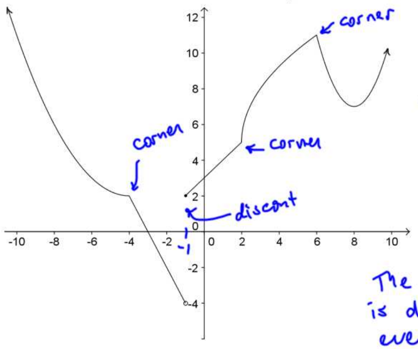
How can the graph of a function be used to determine where a function is not differentiable?

A function is not differentiable at:

1. Points of discontinuity.
2. Points where the graph has a corner.
3. Points where the graph has a cusp. $y = x^{2/3}$ at $x=0$
4. Points where the graph has a vertical tangent.

$y = x^{1/3}$ at $x=0$

The function $f(x)$ is graphed below. Determine the values of x where f is differentiable.



Problems

$x=-4$

$x=-1$

$x=2$

$x=6$

The function is diff. everywhere else.

$\therefore$ f is differentiable on

$(-\infty,-4)\cup(-4,-1)\cup(-1,2)\cup(2,6)\cup(6,\infty)$.