Test 2 Review

- Evaluating limits of basic functions, from formulas and graphs.
- Evaluating $\sin(u)/u$ type limits.
- Determining continuity of functions from formulas and graphs.
- Intermediate value theorem.
- Using the definition of derivative to find a derivative.
- Using a graph to determine where a derivative does not exist.
- Using the power, product and quotient rules to find derivatives.
- Derivatives of trigonometric functions.
- Chain rule.
- Tangent and normal lines.
- Implicit differentiation.
- Rates of change and related rates.

Test 2 will not contain any problems related to $\varepsilon\delta$ proofs, the extreme value theorem or the pinching theorem.

Good Sources of Practice Problems

- The textbook
- Homework
- Poppers
- EMCFs
- Online Quizzes
- Online Pratice Test 2
- Lab/workshop quizzes
- Class examples
- This problem set
- The other posted review set
- Next Tuesday's review

Examples: $\lim_{x\to 3}(3x^2-5x+2) = 3\cdot 9 - 5\cdot 3 + 2 = 14$

$$\lim_{x\to -1}\frac{\sin(3x)}{2x} = \frac{\sin(-3)}{-2} = \frac{-\sin(3)}{-2} = \frac{\sin(3)}{2}$$

$$\lim_{x\to 1}\frac{x^2-3x-2}{x^2-1} = \text{DNE}$$

"$\frac{-4}{0}$"

b/c the numerator goes to a nonzero value and the denominator goes to 0.

Examples: $\lim_{x\to 2}\frac{x^2-4}{x+2} = \frac{0}{4} = 0$

"$\frac{0}{0}$" ind form

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 4$$

"$\frac{0}{0}$" ind form

$$\lim_{x\to 0}\frac{\sin(2x)}{\sin(3x)} = \lim_{x\to 0}\frac{2}{3}\boxed{\frac{\sin(2x)}{2x}}\cdot\boxed{\frac{3x}{\sin(3x)}} = \frac{2}{3}$$

"$\frac{0}{0}$" ind form

$$\lim_{x\to 0}\frac{x^2}{1-\cos(x)} = \lim_{x\to 0}\frac{x^2(1+\cos(x))}{(1-\cos(x))(1+\cos(x))}$$

$$= \lim_{x\to 0}\frac{x^2}{\sin^2(x)}(1+\cos(x))$$

$$= \lim_{x\to 0}\Big(\underbrace{\frac{x}{\sin(x)}}_{\to 1}\Big)^2\underbrace{(1+\cos(x))}_{\to 2} = 2$$

Example: Give values of A and B so that $g(x)=\begin{cases} Ax+B, & x<2 \\ 2, & x=2 \\ Ax^2-Bx, & x>2 \end{cases}$ is continuous.

The only possible issue is $x=2$.
Continuity at $x=2$ requires

$$\underset{=\,2}{g(2)} = \lim_{x\to 2} g(x)$$

$$\lim_{x\to 2^-}(Ax+B)=2A+B$$

$$\lim_{x\to 2^+}(Ax^2-Bx)=4A-2B$$

$$\begin{aligned} 2\,(2A+B&=2) \\ +\ (4A-2B&=2) \\ \hline 8A&=6 \Rightarrow A=3/4 \end{aligned}$$

Subst. into $4A-2B=2 \Rightarrow 3-2B=2$
$\Rightarrow 2B=1 \Rightarrow B=1/2$

$$\boxed{A=3/4,\ B=1/2}$$

Example: Describe the discontinuities of the function $f(x)=\underbrace{\dfrac{x+2}{x^2+4x+4}}_{\text{rational function}}$.

Note: since f is a rational function, f is continuous everywhere it is defined.

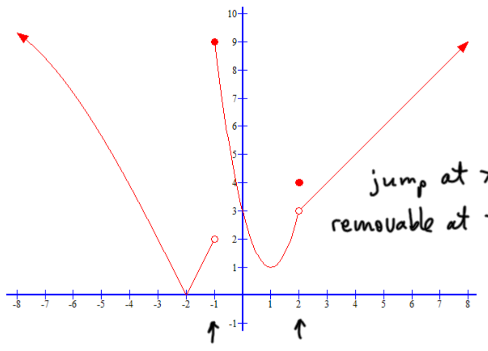
f is not defined when $x^2+4x+4=0$.
$(x+2)^2=0$

$\therefore$ The only discontinuity is at $x=-2$.

Note: $f(x)=\dfrac{x+2}{(x+2)^2}=\dfrac{1}{x+2}$

this has a vertical asymptote at $x=-2$. $\therefore f$ has an infinite discontinuity at $x=-2$.

Example: The function f is shown below. Describe any discontinuities.



jump at $x=-1$
removable at $x=2$

Example: Use the intermediate value theorem to prove there is a solution to $\underbrace{3x^3-2x^2-2x}_{\text{polynomial}}=1$ on the interval $[-1,3]$.

$f(x)=3x^3-2x^2-2x$ is continuous on $[-1,3]$.

Note: $f(-1)=-3-2+2=-3$
$f(3)=81-18-6=81-24$
$=57$

1 is between $f(-1)$ and $f(3)$

$\therefore$ there is a value c between -1 and 3 so that $f(c)=1$. } From IVThm

Examples: $\frac{d}{dx}\left(\frac{2x-1}{3x^2+4x}\right) = \frac{(3x^2+4x)\cdot 2 - (2x-1)(6x+4)}{(3x^2+4x)^2}$

$\frac{d}{dx}(\sin(3x)\cos(2x)) = \sin(3x)(-2\sin(2x)) + \cos(2x)\cdot\cos(3x)\cdot 3$

$= -2\sin(3x)\sin(2x) + 3\cos(2x)\cos(3x)$

$f(x)=2(x^2+1)^3$. Give $f''(x)$.

$f'(x) = 6(x^2+1)^2\cdot 2x = 12x(x^2+1)^2$

Example: $G(x)=(f\circ g)(x)$. Suppose $f(2)=3$, $f'(2)=3$, $f(1)=-2$, $f'(1)=-4$, $g(1)=2$, and $g'(1)=-3$. Give $G'(1)$.

$G(x) = f(g(x))$

1. $G'(x) = f'(g(x))\,g'(x)$

2. $G'(1) = f'(g(1))\,g'(1)$

$= f'(2)(-3)$

$= 3(-3) = -9$

Example: $G(x)=\sin(\pi f(x)) + (2g(x)-1)^2$. Suppose $f(2)=3$, $f'(2)=3$, $f(1)=-2$, $f'(1)=-4$, $g(1)=2$, and $g'(1)=-3$. Give $G'(1)$.

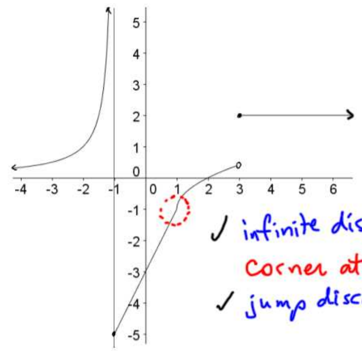
1. $G'(x) = \cos(\pi f(x))\pi f'(x) + 2(2g(x)-1)\cdot 2g'(x)$

2. $G'(1) = \cos(\pi f(1))\cdot \pi f'(1) + 2(2g(1)-1)\cdot 2g'(1)$

$= \cos(-2\pi)\cdot(-4\pi) + 2(4-1)\cdot 2(-3)$

$= -4\pi - 36$

Example: The function f is shown below. Describe any discontinuities. Also, give the values where the function is not differentiable.



✓ infinite discont at $x=-1$

Corner at $x=1$

✓ jump discont at $x=3$

f is not diff at $x=-1$, $x=1$ and $x=3$.

$g(x) = \dfrac{a}{bx+c}$, $g(x) = ax^2+bx+c$, $g(x) = \sqrt{ax+b}$

Example: Use the definition of derivative to find the derivative of $f(x) = \dfrac{3}{2x-5}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{2(x+h)-5} - \frac{3}{2x-5}}{h}$$

"$\frac{0}{0}$" ind form

$$= \lim_{h\to 0} \frac{\frac{6x-15-(6(x+h)-15)}{(2(x+h)-5)(2x-5)}}{h}$$

$$= \lim_{h\to 0} \frac{-6\cancel{h}}{\cancel{h}(2(x+h)-5)(2x-5)}$$

$$= \frac{-6}{(2x-5)^2}$$

Example: Give a formula for $\frac{dy}{dx}$ in terms of x and y, given that $x^3 - 3xy + 2y^3 = 2$. Then give the equation of the normal line to the graph at the point $(0,1)$.

Treat y like a diff. function of x, and diff both sides wrt x.

$$\frac{d}{dx}(x^3 - 3xy + 2y^3) = \frac{d}{dx}2$$

$$3x^2 - \left[3x\frac{dy}{dx} + y\cdot 3\right] + 6y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(-3x + 6y^2) = -3x^2 + 3y$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-x^2+y}{-x+2y^2}}$$

N.L. at $(0,1)$: slope $= \dfrac{-1}{\left.\frac{dy}{dx}\right|_{(0,1)}} = \dfrac{-1}{1/2} = -2$

Point: $(0,1)$

Equation: $y - 1 = -2x$

or $y = -2x + 1$

Example: An object is thrown upwards with an initial velocity of 20 ft/sec, and it strikes the ground 5 seconds later. Give the initial height of the object, and the speed of the object on impact. (Neglect air friction.)

Falling body problem.

$$s(t) = -16t^2 + v_0 t + h_0$$

$v_0 = 20$, $h_0 = s_0$

$S(5) = 0$ $-16\cdot 25 + 100 + h_0 = 0$

$-400 + 100 + h_0 = 0$

$\boxed{h_0 = 300 \text{ ft}} = s_0$

speed on impact $= |v(5)| = |s'(5)|$

$v(t) = s'(t) = -32t + 20$ $= |-160 + 20|$

$= 140$ ft/sec.

Example: A particle moves along the real line so that its position at time t seconds is given by $x(t) = t - 3t^2 + \cos(2\pi t)$ feet. Give the velocity and speed of the particle when $t = 1/3$.

$$v(\tfrac{1}{3}) = x'(1/3)$$

$$[\text{speed at } t = 1/3] = |x'(1/3)|$$

$$x'(t) = 1 - 6t - 2\pi\sin(2\pi t)$$

$$x'(1/3) = 1 - 2 - 2\pi\sin\left(\tfrac{2\pi}{3}\right)$$

$$= -1 - 2\pi\frac{\sqrt{3}}{2} = -1 - \pi\sqrt{3} \text{ ft/sec}$$

i.e. $[\text{velocity at } t = \tfrac{1}{3}] = -1 - \pi\sqrt{3}$ ft/sec.

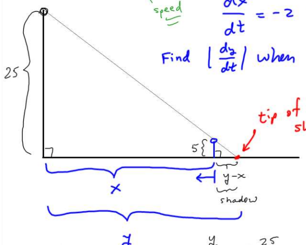
$[\text{speed at } t = 1/3] = 1 + \pi\sqrt{3}$ ft/sec.

Example: A 5 foot tall girl is walking towards a 25 foot lamp post at the rate of 2 feet per second. How fast is the tip of her shadow moving when she is 10 feet from the lamp post?

speed

$\frac{dx}{dt} = -2$ ft/sec.

Find $\left|\frac{dy}{dt}\right|$ when $x = 10$.



similar triangles $\quad \frac{y}{y-x} = \frac{25}{5}$

$\Rightarrow \frac{y}{y-x} = 5 \quad \Rightarrow \quad y = 5y - 5x$

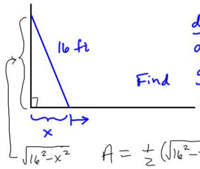
$4y = 5x$

Diff wrt t. $\quad 4\frac{dy}{dt} = 5\frac{dx}{dt}$

Find $\left|\frac{dy}{dt}\right|$ $\qquad$ -2

$\Rightarrow \frac{dy}{dt} = -\frac{5}{2} \quad \Rightarrow \left|\frac{dy}{dt}\right| = \frac{5}{2}$ ft/sec.

Example: A 16 foot board is leaning against a vertical wall. If the bottom of the board slides away from the wall at the rate of 3 feet per second, how fast is the area of the triangle formed by the board, the floor and the wall changing at the instant when the bottom of the board is 8 feet from the wall?



$\frac{dx}{dt} = 3$ ft/sec.

Find $\frac{dA}{dt}$ when $x = 8$.

$\sqrt{16^2 - x^2} \qquad A = \frac{1}{2}\left(\sqrt{16^2 - x^2}\right)x$

$\Rightarrow A = \frac{1}{2}x\sqrt{16^2 - x^2}$

$A = \frac{1}{2}x\sqrt{256 - x^2}$

Diff wrt t.

$$\frac{dA}{dt} = \frac{1}{2}x\frac{d}{dt}\sqrt{256-x^2} + \sqrt{256-x^2}\cdot\frac{d}{dt}\frac{1}{2}x$$

$$\frac{dA}{dt} = \frac{1}{2}x\cdot\frac{1}{2\sqrt{256-x^2}}\cdot\left(-2x\frac{dx}{dt}\right) + \sqrt{256-x^2}\left(\frac{1}{2}\frac{dx}{dt}\right)$$

Find when $x=8$ $\qquad$ 3 $\qquad$ 3

$= \dots = \boxed{??}$ ft²/sec

you do it.