Info

- Test 2 starts this week.
- EMCTs have been posted for the week.
- There is no homework due next Monday.
- There will be an EMCF due next Monday.
- There will be a quiz in lab on Friday.
- There is an online review tomorrow night.

Differentials and Newton's Method

(applications of tangent lines)

Newton's Method - Geometrically

Goal: Approximate a solution to \( f(x) = 0 \).

- Target line at \( x_0 \):
  \[ y - f(x_0) = \frac{f(x_0)}{f'(x_0)}(x - x_0) \]
- Solve for \( x \):
  \[ x = x_0 - \frac{f(x_0)}{f'(x_0)} \]
- Update:
  \[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

Newton's Method - Formula

Let \( f \) be a twice differentiable function and suppose \( a \) is a real number at which \( f(a) = 0 \).
If \( f'(a) \neq 0 \) and \( x_0 \) is sufficiently close to \( a \), then the iteration
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
will converge (rapidly) to the root \( a \).
Example: Use several iterations of Newton's method from a guess of $x = 1$ to approximate a solution to $x^2 = x - 1 = 0$.

\[
\begin{align*}
\frac{d}{dx}(x^2 - x + 1) &= 0, \\
\frac{d^2}{dx^2}(x^2 - x + 1) &= 1 \\
\frac{d^3}{dx^3}(x^2 - x + 1) &= 2x - 1 = 0 \\
\frac{d^4}{dx^4}(x^2 - x + 1) &= 2 \\
\end{align*}
\]

\[
\begin{align*}
x_0 &= 1, \\
x_1 &= x_0 \cdot \frac{f(x_0)}{f'(x_0)} = 1 \cdot \frac{1}{2} = 0.5, \\
x_2 &= x_1 \cdot \frac{f(x_1)}{f'(x_1)} = 0.5 \cdot \frac{1}{2} = 0.25, \\
x_3 &= x_2 \cdot \frac{f(x_2)}{f'(x_2)} = 0.25 \cdot 2 = 0.5, \\
x_4 &= x_3 \cdot \frac{f(x_3)}{f'(x_3)} = 0.5 \cdot 2 = 1, \\
\end{align*}
\]

Notice that we don’t see any improvement after the $x_3$ value. You can check that $f(x)$ is

VERY CLOSE to zero!!
Definition: The differential of \( f \) at \( a \) with increment \( h \) is given by
\[
df = f'(a) \, h
\]

Differentials Can Be Used To Approximate Function Values

The differential of \( f \) at \( a \) with increment \( h \) is given by \( df = f'(a)h \).

Using the approximation \( df \approx f(a+h) - f(a) \), the equation above becomes
\[
f(a+h) = f(a) + f'(a)h
\]
(this is a tangent line approximation)

Example: Give the differential of \( f(x) = \sqrt{x} \) at \( x = 25 \) with increment 0.1.

\[
\begin{align*}
f'(x) &= \frac{1}{2\sqrt{x}} \\
f'(25) &= \frac{1}{2 \cdot 5} = \frac{1}{10} \\
df &= f'(25) \cdot 0.1 \\
&= \frac{1}{10} \cdot 0.1 = \frac{1}{100}
\end{align*}
\]

Example: Use differentials to approximate \( \sqrt{25.1} \).

\[
\begin{align*}
f(x+h) &= f(x) + df \\
g(x+h) &= \sqrt{x+1} \\
\sqrt{25} &= f(5) \\
df &= f'(5) \cdot 1 \\
&= \frac{1}{10} \\
&= 0.1
\end{align*}
\]

Correct: \( \sqrt{25.1} \approx 5 + 0.1 = 5.1 \)

(Note: \( \sqrt{25.1} \approx 5.00999802 \))
Example: A box is to be constructed in the form of a cube to hold 1000 cubic feet. Use a differential to estimate how accurately the edge must be made so that the volume will be correct to within 3 cubic feet.

Next Time!!