Info

- Test 2 starts this week.
- EMCFs have been posted for the week.
- There is no homework due *next* Monday.
- There will be an EMCF due *next* Monday.
- There will be a quiz in lab on Friday.
- There is an online review tomorrow night.

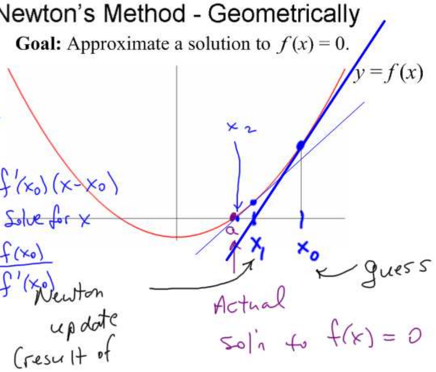
Differentials and Newton's Method

Section 3.9

(applications of tangent lines)

Newton's Method - Geometrically

Goal: Approximate a solution to $f(x) = 0$.



Tangent line at x_0.

$y - f(x_0) = f'(x_0)(x - x_0)$

Set $y = 0$. Solve for x

$x = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

x_1

Newton update (result of on newton iteration)

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ ...

Newton's Method - Formula

Let f be a twice differentiable function and suppose a is a real number at which $f(a) = 0$. If $f'(a) \neq 0$ and x_0 is sufficiently close to a, then the iteration

Guess

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

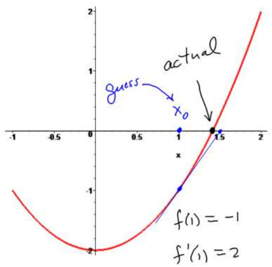
will converge (rapidly) to the root a.

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$\vdots$

Example: Use 2 iterations of Newton's method to approximate $\sqrt{2}$, starting from a guess of 1. Hint: Use the function $f(x) = x^2 - 2$.



$x^2 - 2 = 0$

Note: $\sqrt{2}$ solves this equation.

$f(x) = x^2 - 2$

$f'(x) = 2x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 1 - \frac{f(1)}{f'(1)} = 1.5$

$f(1) = -1$

$f'(1) = 2$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(\frac{3}{2}) = \frac{1}{4}$

$f'(\frac{3}{2}) = 3$

$= \frac{3}{2} - \frac{f(3/2)}{f'(3/2)}$

$= \frac{3}{2} - \frac{1/4}{3} = \frac{3}{2} - \frac{1}{12}$

$1.41\overline{6} \longrightarrow = \frac{17}{12}$

i	xi
0	1
1	1.5
2	1.416666667
3	1.414215686
4	1.414213562
5	1.414213562
6	1.414213562
7	1.414213562
8	1.414213562
9	1.414213562
10	1.414213562

Snapshot of the process using Excel. Notice that the 4th iterate is excellent, and we do not see any improvement (here) after that point.

$\sqrt{2} = 1.414213562\ldots$

Example: Use several iterations of Newton's method from a guess of $x = 1$ to approximate a solution to $x^4 + x - 1 = 0$.

x_0

$f(x) = x^4 + x - 1$

$f'(x) = 4x^3 + 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{5}$

$= .8$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = .8 - \frac{f(.8)}{f'(.8)} = .7312335958$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \ldots$

i	xi
0	1
1	0.8
2	0.731233596
3	0.72454848
4	$x_4 =$ 0.724491963
5	$x_5 =$ 0.724491959
6	0.724491959
7	0.724491959
8	0.724491959
9	0.724491959
10	0.724491959

Notice that we don't see any improvement after the x_5 value. You can check that $f(x_5)$ is VERY CLOSE to zero!!

Example: Use Newton's Method to develop an algorithm for giving excellent approximations to square roots.

$a > 0$ Goal: Approx $\sqrt{a}$.

Use the equation $x^2 - a = 0$

$f(x) = x^2 - a$

$f'(x) = 2x$

Guess: $x_0 = 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - a}{2x_0}$

$= x_0 - \frac{x_0}{2} + \frac{a}{2x_0}$

Converges rapidly to $\sqrt{a}$.

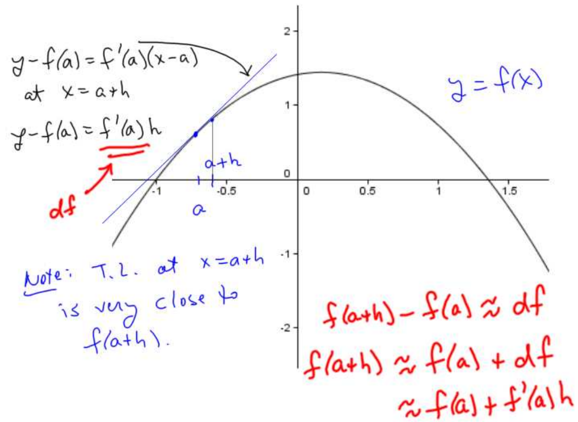
$$\begin{cases} x_1 = \frac{x_0}{2} + \frac{a}{2x_0} \\ x_2 = \frac{x_1}{2} + \frac{a}{2x_1} \\ x_3 = \frac{x_2}{2} + \frac{a}{2x_2} \\ \vdots \end{cases}$$

Definition: The differential of f at a with increment h is given by

$$df = f'(a)\,h$$

differential of f at $x=a$ with increment h.

Geometric Interpretation



Note: T.L. at $x=a+h$ is very close to $f(a+h)$.

$f(a+h) - f(a) \approx df$

$f(a+h) \approx f(a) + df$

$\approx f(a) + f'(a)h$

Differentials Can Be Used To Approximate Function Values

def: The differential of f at a with increment h is given by $df = f'(a)h$

Using the approximation $df \approx f(a+h) - f(a)$, the equation above becomes

$$f(a+h) \approx f(a) + f'(a)h$$

(this is a tangent line approximation)

Example: Give the differential of $f(x) = \sqrt{x}$ at $x = 25$ with increment 0.1.

h, a

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$df = f'(25)\cdot(0.1) = \frac{1}{10}\cdot\frac{1}{10} = \frac{1}{100}$$

Example: Use differentials to approximate $\sqrt{25.1}$.

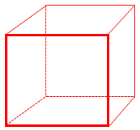
$f(x) = \sqrt{x}$
$a = 25$
$a+h = 25.1$
$h = .1$

$$f(a+h) \approx f(a) + \underbrace{f'(a)h}_{df}$$

$$\sqrt{25.1} = f(25.1) \approx f(25) + df = 5 + \frac{1}{100} = 5.01$$

Note: $\sqrt{25.1} = 5.00999002$

Example: A box is to be constructed in the form of a cube to hold 1000 cubic feet. Use a differential to estimate how accurately the edge must be made so that the volume will be correct to within 3 cubic feet.



Next Time!!