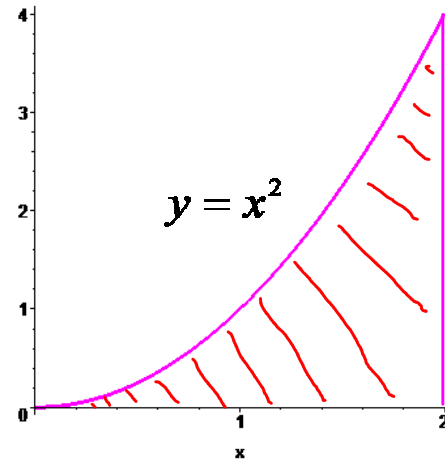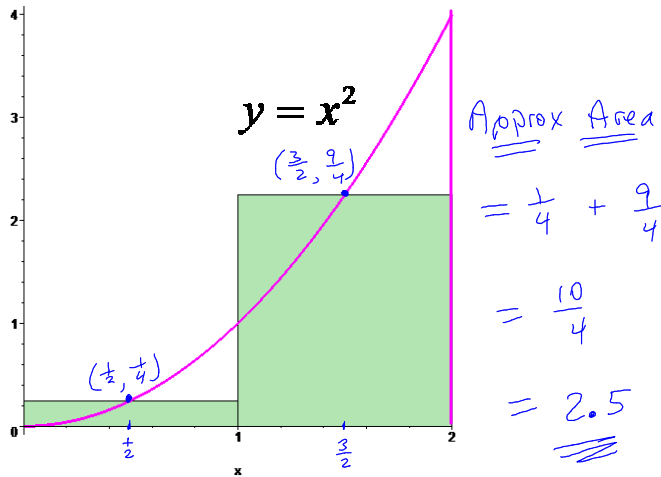## Area and Integrals

**1. Test 3 has nearly started.**
2. The notes and video are posted from last night's review.
3. There is no Homework due on Monday. There will be an EMCF due on Monday and a Quiz due Monday night.
3. Practice Test 3 is due very soon!
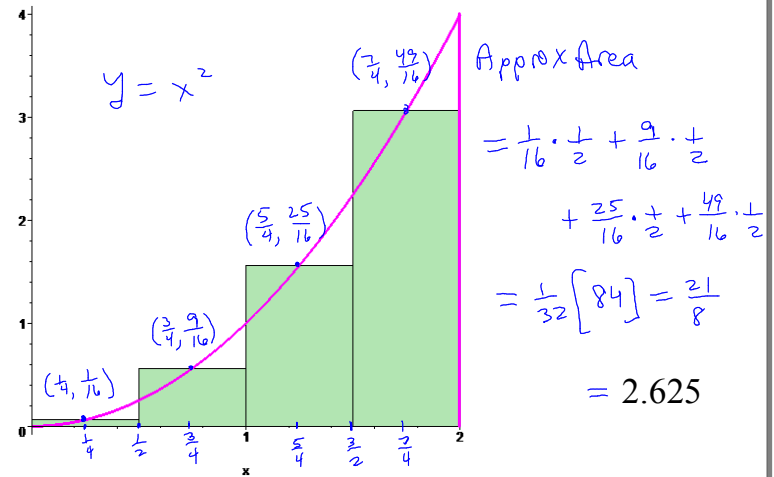4. We are starting chapter 5 today.

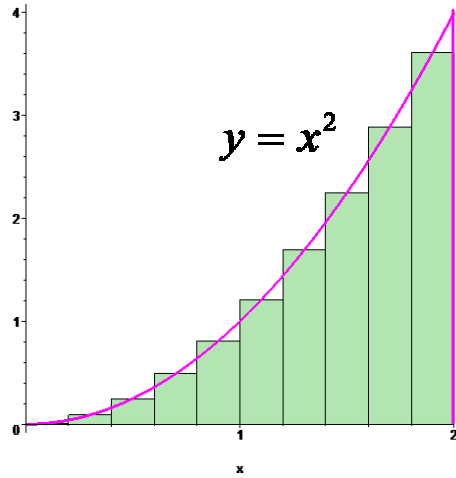**Question:** Is there a method for approximating the area below?



$y = x^2$

We could approximate the area with 2 rectangles.



$y = x^2$

$\left(\frac{3}{2}, \frac{9}{4}\right)$

$\left(\frac{1}{2}, \frac{1}{4}\right)$

Approx Area

$= \frac{1}{4} + \frac{9}{4}$

$= \frac{10}{4}$

$= 2.5$

We could approximate the area with 4 rectangles.



$y = x^2$

$\left(\frac{7}{4}, \frac{49}{16}\right)$

$\left(\frac{5}{4}, \frac{25}{16}\right)$

$\left(\frac{3}{4}, \frac{9}{16}\right)$

$\left(\frac{1}{4}, \frac{1}{16}\right)$

Approx Area

$= \frac{1}{16} \cdot \frac{1}{2} + \frac{9}{16} \cdot \frac{1}{2}$

$+ \frac{25}{16} \cdot \frac{1}{2} + \frac{49}{16} \cdot \frac{1}{2}$

$= \frac{1}{32}\left[84\right] = \frac{21}{8}$
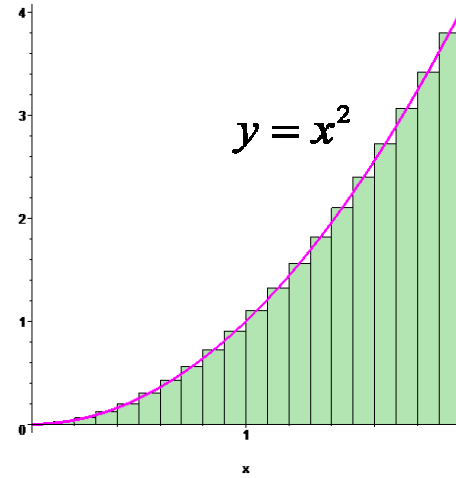
$= 2.625$

**1**

We could approximate the area with 10 rectangles.

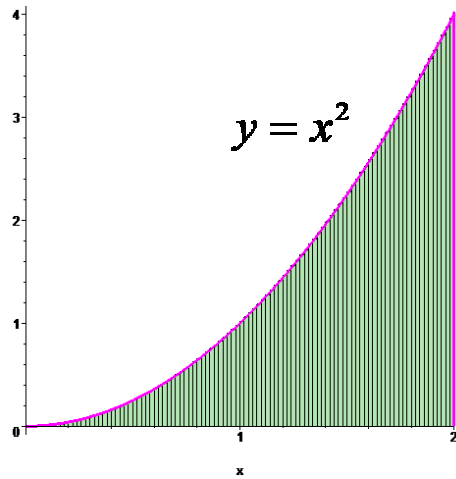

Crude, but better.

Approximate area
2.66

We could approximate the area with 20 rectangles.
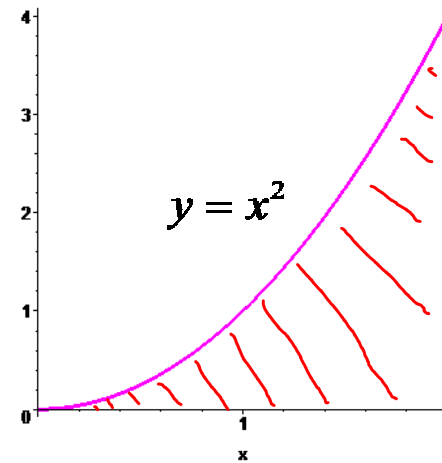


Much better.

Approximate area
2.665

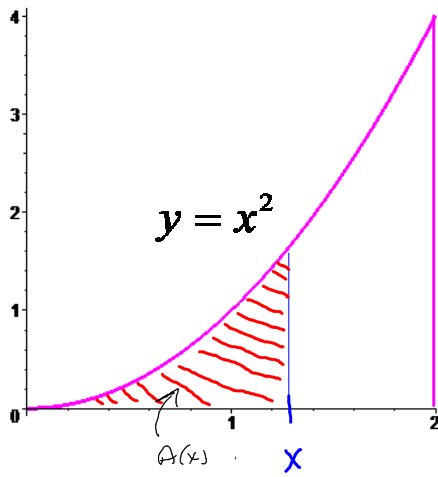We could approximate the area with 100 rectangles.



Much much better.

Approximate area
2.6666

Let's Find the Exact Value for this Area



2

**Step 1:** Define the function A($x$) that gives the area shown below from 0 to $x$.
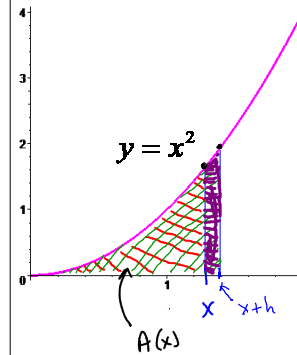


**What is A(0) ?**

**What is A(2) ?** The area we want.

We will get a formula for the function A($x$).

$y = x^2$

A($x$)   **X**

---

**Step 2:** Compute the derivative of A($x$).



$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

$y = x^2$

like a trapezoid for small $h$.

$$= \lim_{h \to 0} \frac{\text{Area}}{h}$$

A($x$)   $x$   $x+h$

$$= \lim_{h \to 0} \frac{\frac{1}{2}\left(x^2 + (x+h)^2\right)h}{h}$$

$$A'(x) = x^2$$

---

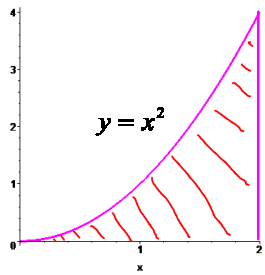**Step 3:** Give a function A($x$) having this derivative, and satisfying A(0) = 0.

$$A'(x) = x^2, \quad \boxed{A(0) = 0}$$

$$A(x) = \frac{1}{3}x^3 + C \longrightarrow 0 = \frac{1}{3}0^3 + C$$

constant

$$\Rightarrow C = 0$$

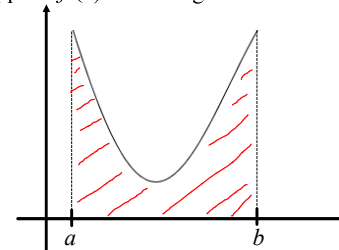$$A(x) = \frac{1}{3}x^3$$

**Step 4:** Determine the original area.

$$\Longleftrightarrow A(2) = \frac{1}{3}8 = \frac{8}{3}$$

$$= 2.6\bar{6}$$

exact



$y = x^2$

---

**Let's generallize this process:**

→ Continuous and

Suppose $f(x)$ is nonnegative on the interval $[a,b]$.



**Goal:** Find the area of the region between the $x$-axis and the graph of $y = f(x)$ over the interval $[a,b]$.

**Computing the area:**

Antiderivative of $f(x)$

1. Find any function $\boxed{F(x)}$ so that $F'(x) = f(x)$.

2. The area is given by $\boxed{F(b) - F(a)}$. $= F(x)\Big|_a^b$

**3**

**Notation:** Suppose $f(x)$ is a continuous function on the interval $[a,b]$. If $F(x)$ is an anti-derivative of $f(x)$, then
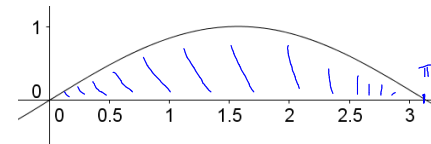
$$\int_a^b f(x)dx = \text{the Riemann Integral of } f(x) \text{ on the interval } [a,b].$$

and

$$\int_a^b f(x)dx = F(b) - F(a)$$

**Note:** If $f(x)$ is nonnegative on the interval $[a,b]$, then the Riemann integral of $f(x)$ on the interval $[a,b]$ is the area bounded between the $x$-axis and the graph of $y = f(x)$ on the interval $[a,b]$.

**Example:** Give the area bounded between the $x$-axis and the graph of $y = \sin(x)$ on the interval $[0,\pi]$.



Note: $\sin(x) \geq 0$ on $[0,\pi]$.

$$\therefore \quad \text{Area} = \int_0^\pi \sin(x)\,dx = \left(\text{anti derivative of } \sin(x)\right)\Big|_0^\pi$$

Note:
$$\frac{d}{dx}\left(-\cos(x)\right) = \sin(x)$$

$$= \left(-\cos(x)\right)\Big|_0^\pi$$
$$= \left(-\cos(\pi)\right) - \left(-\cos(0)\right)$$
$$= 1 + 1 = 2$$

**Example:** Give the values for

$$\int_{-1}^{2}\left(x^2 + x\right)dx, \quad \int_{1}^{3}\left(2\sqrt{x} + \frac{1}{x^2}\right)dx$$

**See the video!!**