Area and Integrals

1. Test 3 has nearly started.
2. The notes and video are posted from last night's review.
3. There is no Homework due on Monday. There will be an EMCF due on Monday and a Quiz due Monday night.
3. Practice Test 3 is due very soon!
4. We are starting chapter 5 today.
**Question:** Is there a method for approximating the area below?

\[ y = x^2 \]
We could approximate the area with 2 rectangles.

\[ y = x^2 \]

\[ \left( \frac{3}{2}, \frac{9}{4} \right) \]

Approximate Area

\[ = \frac{1}{4} + \frac{9}{4} \]

\[ = \frac{5}{2} \]

\[ = 2.5 \]
We could approximate the area with 4 rectangles.

Let's calculate the area:

\[ y = x^2 \]

Approx. Area

\[ = \frac{1}{16} \cdot \frac{1}{2} + \frac{9}{16} \cdot \frac{1}{2} + \frac{25}{16} \cdot \frac{1}{2} + \frac{49}{16} \cdot \frac{1}{2} \]

\[ = \frac{1}{2} \cdot \frac{84}{16} = \frac{42}{16} = 2.625 \]
We could approximate the area with 10 rectangles.

Crude, but better.

$y = x^2$

Approximate area

2.66
We could approximate the area with 20 rectangles.  

\[ y = x^2 \]

Much better.  
Approximate area  
2.665
We could approximate the area with 100 rectangles.

\[ y = x^2 \]

Much much better.

Approximate area
2.6666
Let's Find the Exact Value for this Area

\[ y = x^2 \]
Step 1: Define the function $A(x)$ that gives the area shown below from 0 to $x$.

What is $A(0)$?

What is $A(2)$?

The area we want.

We will get a formula for the function $A(x)$.
Step 2: Compute the derivative of $A(x)$.

Amazing!

$A'(x) = x^2$
Step 3: Give a function $A(x)$ having this derivative, and satisfying $A(0) = 0$.

\[ A'(x) = x^2, \quad A(0) = 0. \]

\[ A(x) = \frac{1}{3}x^3 + C \]

\[ A(x) = \frac{1}{3}x^3 \]

Step 4: Determine the original area.

\[ y = x^2 \]

\[ A(2) = \frac{1}{3} \cdot 8 \]

\[ = \frac{8}{3} \]

\[ = 2.6 \]
Let’s generalize this process:

Suppose $f'(x)$ is nonnegative on the interval $[a,b]$.

Goal: Find the area of the region between the $x$-axis and the graph of $y = f(x)$ over the interval $[a,b]$.

Computing the area:

1. Find any function $\boxed{F(x)}$ so that $F'(x) = f(x)$.

2. The area is given by $F(b) - F(a)$. 

An antiderivative of $f(x)$. 
**Notation:** Suppose \( f(x) \) is a continuous function on the interval \([a,b]\). If \( F(x) \) is an anti-derivative of \( f(x) \), then

\[
\int_{a}^{b} f(x) \, dx = \text{the Riemann Integral of } f(x) \text{ on the interval } [a,b].
\]

and

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a) = \left. F(x) \right|_{a}^{b}
\]

**Note:** If \( f(x) \) is nonnegative on the interval \([a,b]\), then the Riemann integral of \( f(x) \) on the interval \([a,b]\) is the area bounded between the \(x\)-axis and the graph of \( y = f(x) \) on the interval \([a,b]\).

**Note:** If \( f(x) \) is NOT nonnegative on the interval \([a,b]\), then this Riemann integral of \( f(x) \) on the interval \([a,b]\) IS NOT an area. It is just a number, and we will see the significance later.
Example: Give the area bounded between the $x$-axis and the graph of $y = \sin(x)$ on the interval $[0, \pi]$.

Note: $\sin(x)$ is non-negative on $[0, \pi]$.

\[ \text{Area} = \int_{0}^{\pi} \sin(x) \, dx \]
\[ = -\cos(x) \bigg|_{0}^{\pi} \]
\[ = (-\cos(\pi)) - (-\cos(0)) \]
\[ = 1 + 1 = 2 \]
Example: Give the values for

\[ \int_{-1}^{2} (x^2 + x) \, dx, \quad \int_{1}^{3} \left(2\sqrt{x} + \frac{1}{x^2}\right) \, dx \]

Note: \( x^2 + x \) is negative on a portion of \([-1, 2]\).

So \( \int_{-1}^{2} (x^2 + x) \, dx \) does not represent an area.

\[
\int_{-1}^{2} (x^2 + x) \, dx = \left. \left( \frac{1}{3} x^3 + \frac{1}{2} x^2 \right) \right|_{-1}^{2}
\]

\[
= \left( \frac{8}{3} + 2 \right) - \left( -\frac{1}{3} + \frac{1}{2} \right)
\]

\[
= \frac{14}{3} - \frac{1}{6} = \frac{27}{6} = \frac{9}{2}.
\]
\[
\int_1^3 \left(2\sqrt{x} + \frac{1}{x^2}\right) \, dx
\]

Note: \( 2\sqrt{x} + \frac{1}{x^2} > 0 \) on
\([1, 3]\).

\[\therefore\]
\[
\int_1^3 (2\sqrt{x} + \frac{1}{x^2}) \, dx \text{ is the area shown.}
\]

\[
\left.\left(\frac{4}{3} x^{3/2} - \frac{1}{x}\right)\right|_{1}^{3}
\]

\[= \left(\frac{4}{3} \cdot 3^{3/2} - \frac{1}{3}\right) - \left(\frac{4}{3} - 1\right)
\]

\[= \text{you} \]