Area and Integrals

1. Test 3 has nearly started.

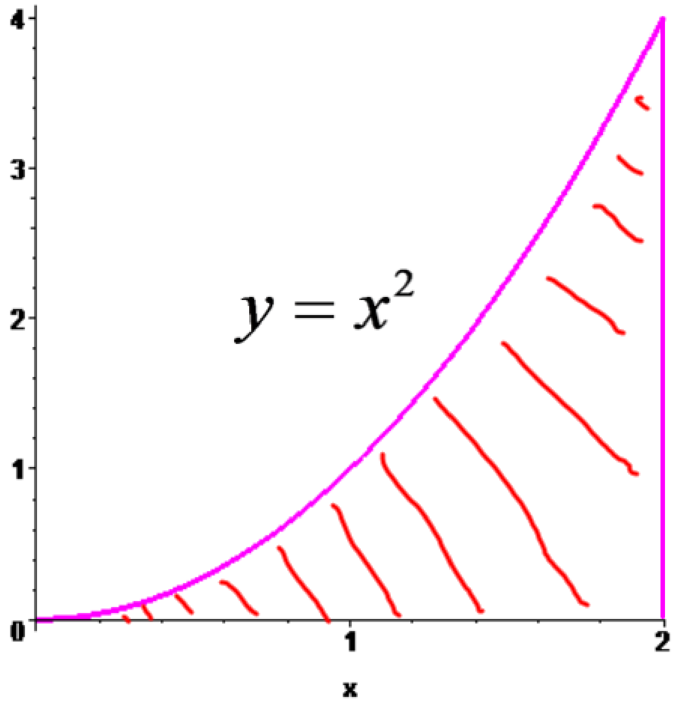
2. The notes and video are posted from last night's review.

3. There is no Homework due on Monday. There will be an EMCF due on Monday and a Quiz due Monday night.

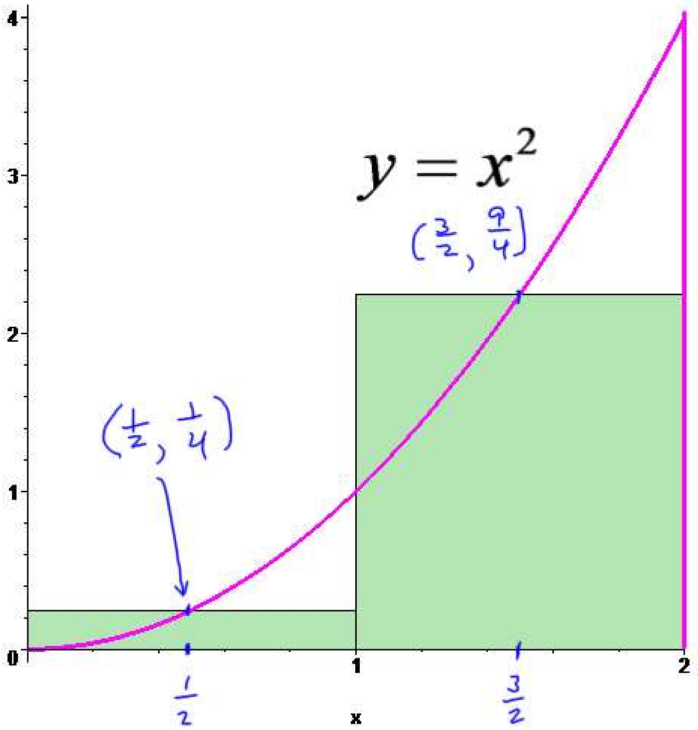
3. Practice Test 3 is due very soon!

4. We are starting chapter 5 today.

Question: Is there a method for approximating the area below?



We could approximate the area with 2 rectangles.



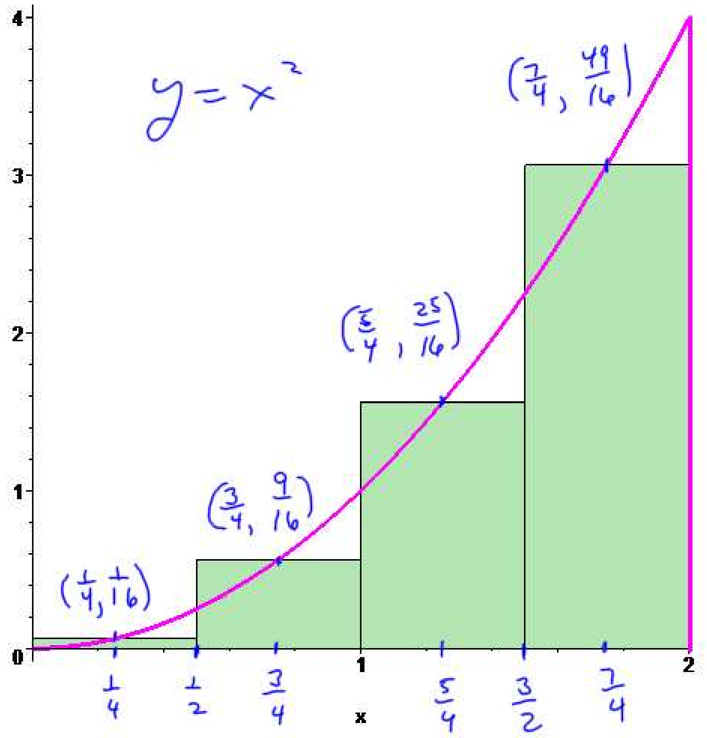
Approx Area

$= \frac{1}{4} + \frac{9}{4}$

$= \frac{5}{2}$

$= \underline{\underline{2.5}}$

We could approximate the area with 4 rectangles.



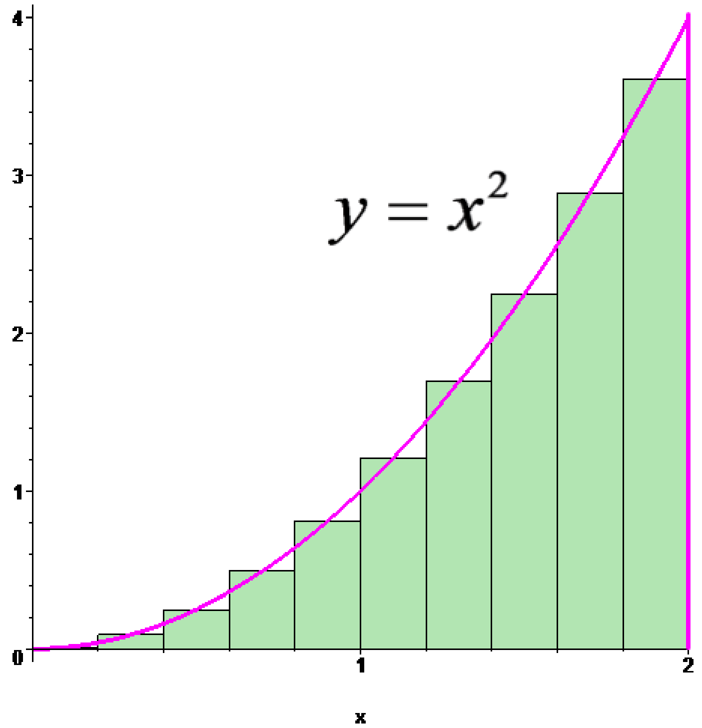
Approx. Area

$$= \frac{1}{16} \cdot \frac{1}{2} + \frac{9}{16} \cdot \frac{1}{2} + \frac{25}{16} \cdot \frac{1}{2} + \frac{49}{16} \cdot \frac{1}{2}$$

$$= \frac{1}{2} \cdot \frac{84}{16} = \frac{42}{16} = \frac{21}{8}$$

$$= 2.625$$

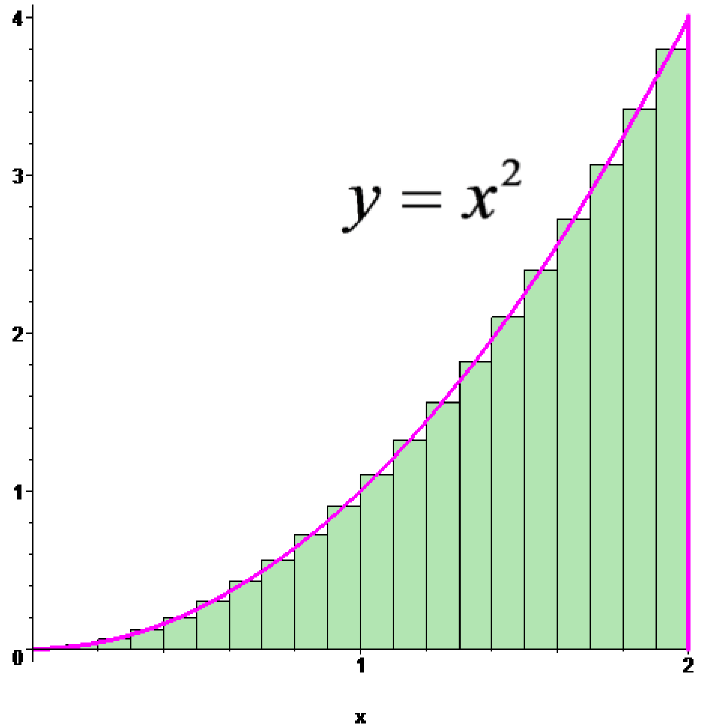
We could approximate the area with 10 rectangles.



Crude, but better.

Approximate
area
2.66

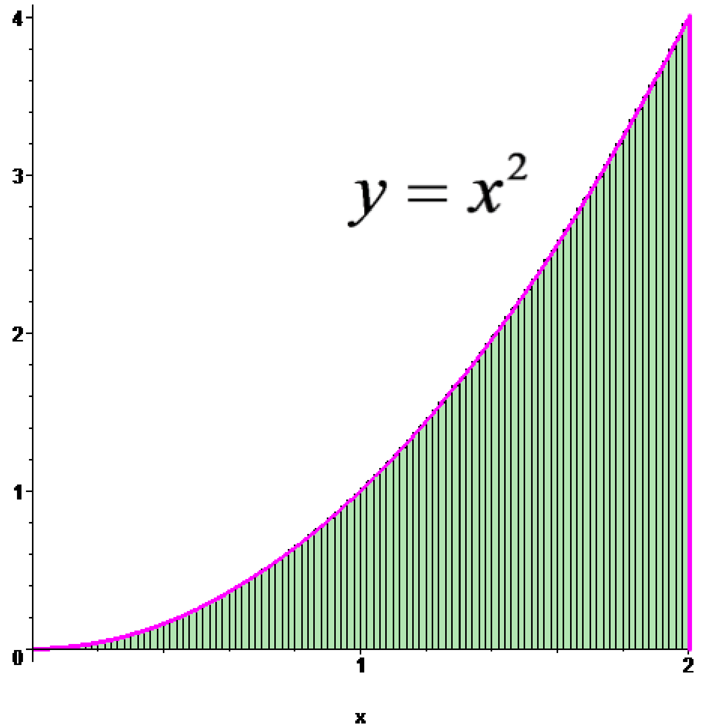
We could approximate the area with 20 rectangles.



Much better.

Approximate area 2.665

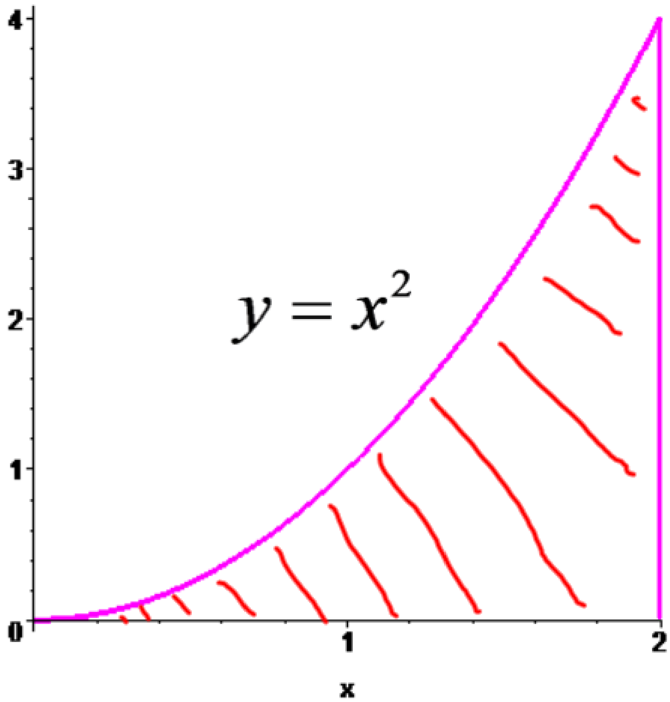
We could approximate the area with 100 rectangles.



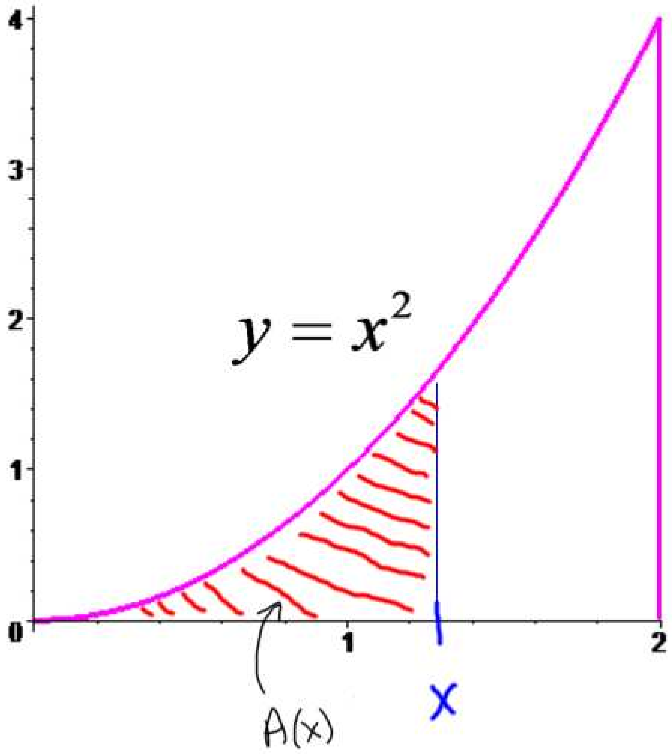
Much much better.

Approximate area 2.6666

Let's Find the Exact Value for this Area



Step 1: Define the function A(x) that gives the area shown below from 0 to x.

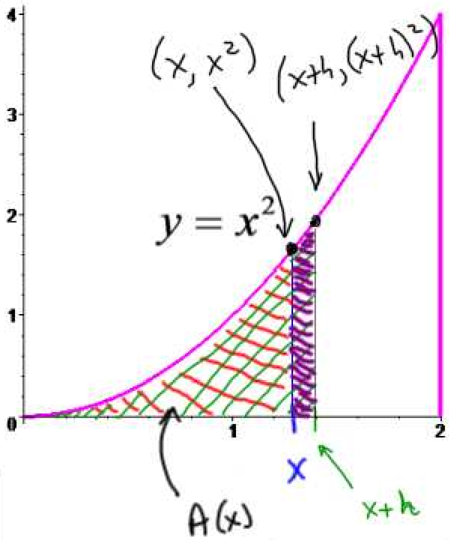


What is A(0) ? 0

What is A(2) ? The area we want.

We will get a formula for the function A(x).

Step 2: Compute the derivative of A(x).



$$A'(x) = \lim_{h\to 0} \frac{\boxed{A(x+h)} - \boxed{A(x)}}{h}$$

nearly a trapezoid

$$= \lim_{h\to 0} \frac{\text{Area}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}\left(x^2 + (x+h)^2\right)\cancel{h}}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{1}{2}\left(x^2 + (x+h)^2\right)$$

$\to x^2$

$$= \frac{1}{2}\cdot 2x^2 = x^2$$

$$\boxed{\text{Amazing! } A'(x) = x^2}$$

Step 3: Give a function A(x) having this derivative, and satisfying A(0) = 0.

$$A'(x) = x^2 \quad , \quad \underline{\underline{A(0)=0}}.$$

$$\hookrightarrow 0 = \frac{1}{3}0^3 + C$$

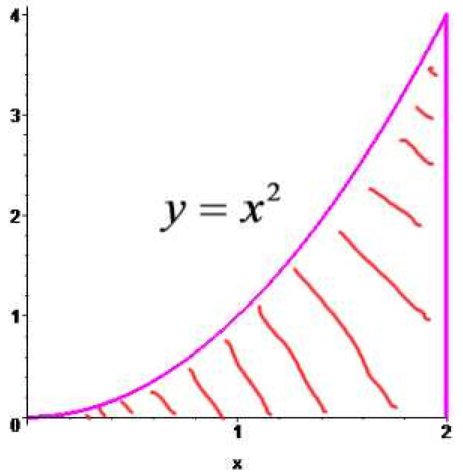
$$\Rightarrow \underline{\underline{C=0}}$$

$$A(x) = \frac{1}{3}x^3 + C$$

constant, to be determined

$$\boxed{A(x) = \frac{1}{3}x^3}$$

Step 4: Determine the original area.



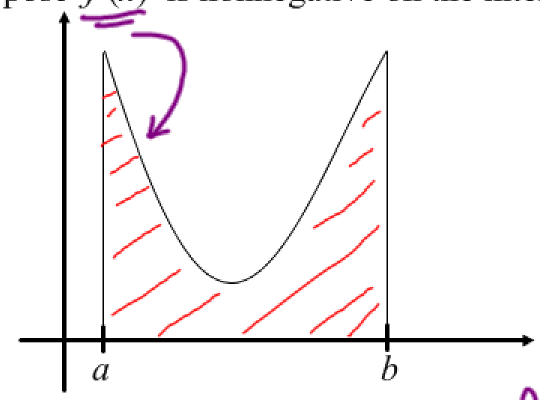
$$A(2) = \frac{1}{3}\cdot 8$$

$$= \frac{8}{3}$$

$$= 2.\overline{6}$$

Let's generallize this process:

Suppose $f(x)$ is nonnegative on the interval $[a,b]$.



Goal: Find the area of the region between the x-axis and the graph of $y = f(x)$ over the interval $[a,b]$.

An antiderivative of $f(x)$.

Computing the area:

1. Find any function $\boxed{F(x)}$ so that $F'(x) = f(x)$.
2. The area is given by $F(b) - F(a)$. $= F(x)\Big|_a^b$

Notation: Suppose $f(x)$ is a continuous function on the interval $[a,b]$. If $F(x)$ is an anti-derivative of $f(x)$, then

$$\int_a^b f(x)dx = \text{ the Riemann Integral of } f(x) \text{ on the interval } [a,b].$$

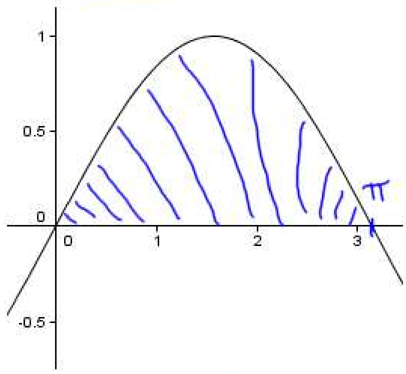
and

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)} = F(x)\Big|_a^b$$

Note: If $f(x)$ is nonnegative on the interval $[a,b]$, then the Riemann integral of $f(x)$ on the interval $[a,b]$ is the area bounded between the x-axis and the graph of $y = f(x)$ on the interval $[a,b]$.

Note: If $f(x)$ is NOT nonnegative on the interval $[a,b]$, then this Riemann integral of $f(x)$ on the interval $[a,b]$ IS NOT an area. It is just a number, and we will see the significance later.

Example: Give the area bounded between the x-axis and the graph of $y = \sin(x)$ on the interval $[0,\pi]$.



Note: sin(x) is nonnegative on $[0, \pi]$.

$$\therefore \text{Area} = \int_0^{\pi} \sin(x)\,dx$$

$$= -\cos(x) \Big|_0^{\pi}$$

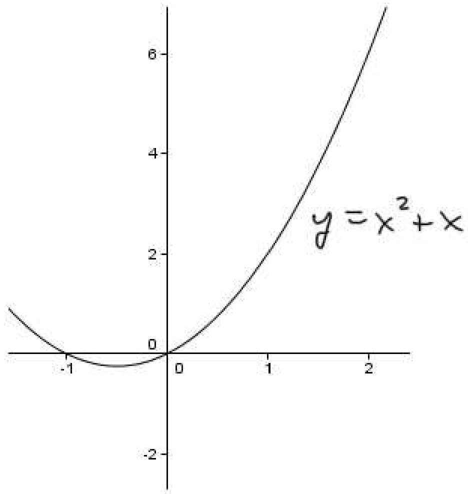
$$= \left(-\cos(\pi)\right) - \left(-\cos(0)\right)$$

$$= 1 + 1 = 2$$

We need an anti-derivative of sin(x). i.e. A function whose derivative is sin(x). $\leftrightarrow -\cos(x)$

Example: Give the values for

$$\int_{-1}^{2}\left(x^2+x\right)dx, \quad \int_{1}^{3}\left(2\sqrt{x}+\frac{1}{x^2}\right)dx$$



Note: x^2+x is negative on a portion of $[-1,2]$.

So $\int_{-1}^{2}(x^2+x)\,dx$ does not represent an area.

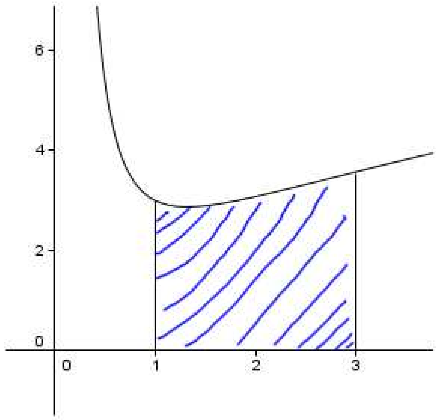
$$\int_{-1}^{2}(x^2+x)\,dx = \left(\text{An antiderivative of } x^2+x\right)\Big|_{-1}^{2}$$

$$= \left(\frac{1}{3}x^3+\frac{1}{2}x^2\right)\Big|_{-1}^{2}$$

$$= \left(\frac{8}{3}+2\right)-\left(-\frac{1}{3}+\frac{1}{2}\right)$$

$$= \frac{14}{3}-\frac{1}{6}=\frac{27}{6}=\boxed{\frac{9}{2}}.$$

$$\int_1^3 \left(2\sqrt{x}+\frac{1}{x^2}\right)dx$$



Note: $2\sqrt{x} + \frac{1}{x^2} > 0$ on $[1, 3]$. $\therefore$

$\int_1^3 (2\sqrt{x} + \frac{1}{x^2})\,dx$ is the area shown.

$$\int_1^3 \left(2\sqrt{x}+\frac{1}{x^2}\right)dx = \left(\text{antiderivative of } 2\sqrt{x} + \frac{1}{x^2}\right)\Big|_1^3$$

$$= \left(\frac{4}{3}x^{3/2} - \frac{1}{x}\right)\Big|_1^3$$

$$= \left(\frac{4}{3}\cdot 3^{3/2} - \frac{1}{3}\right) - \left(\frac{4}{3} - 1\right)$$

$= \text{you.}$ #