Net Area and Integrals

- We are in chapter 5.
- New EMCFs and Homework will be posted.
- Test 3 runs through Monday.
- You have an online quiz and a practice test due Monday.

Review: The first part of the **Fundamental Theorem of Calculus**

Theorem: If f is a continuous function on the interval $[a,b]$, and F is an anti derivative of f, then

$$\underbrace{\int_a^b \boxed{f(x)}\,dx}_{\text{Riemann integral of } f(x) \text{ on } [a,b].} = F(x)\Big|_a^b = F(b) - F(a)$$

integrand

$F'(x) = f(x)$

integral sign

limits of integration

Riemann integral of $f(x)$ on $[a,b]$.

Example: Compute $\int_{-1}^{2} \underbrace{\left(2x^2 - 3x - 5\right)}_{\text{continuous on } [-1, 2].} dx.$

we need an anti derivative

$F(x)$ of $2x^2 - 3x - 5$.

i.e. $F'(x) = 2x^2 - 3x - 5$

$F(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 - 5x$

$\therefore \int_{-1}^{2} (2x^2 - 3x - 5)\,dx = \left(\frac{2}{3}x^3 - \frac{3}{2}x^2 - 5x\right)\Big|_{-1}^{2}$

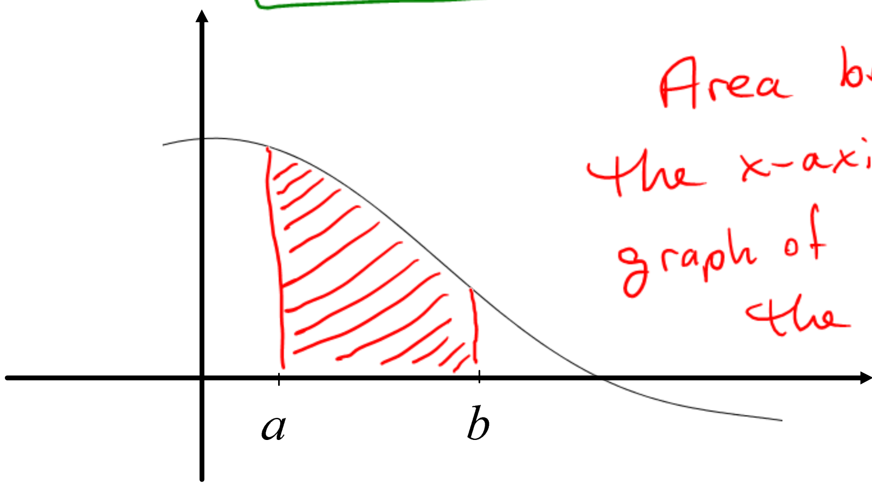
$= \left(\frac{16}{3} - 6 - 10\right) - \left(-\frac{2}{3} - \frac{3}{2} + 5\right)$

$=$ you ...

Review:

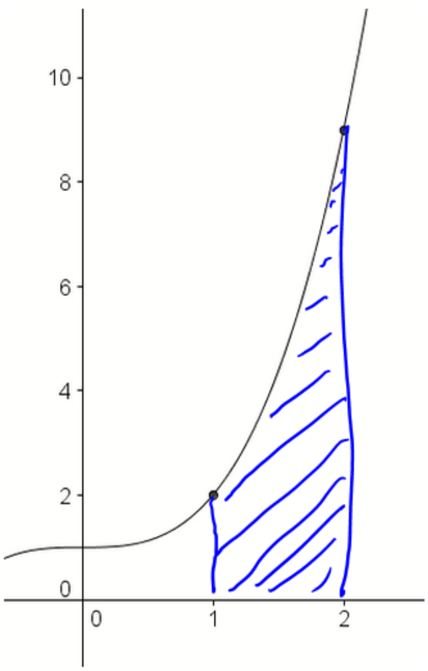
The **Riemann** Integral of f from a to b: $\boxed{\int_a^b f(x)dx}$

1. What do we get if f is nonnegative on the interval $[a,b]$?



Area bounded between the x-axis and the graph of f(x) over the interval [a,b].

Example: Give the area bounded between the x-axis and the graph of $y = x^3 + 1$ on the interval $[1,2]$.



Note: $f(x) = x^3 + 1$ is nonnegative on $[1,2]$.

$$\text{Area} = \int_1^2 (x^3+1)\,dx = \left(\tfrac{1}{4}x^4 + x\right)\Big|_1^2$$

$$= (4+2) - \left(\tfrac{1}{4}+1\right)$$

$$= 5 - \tfrac{1}{4}$$

$$= \tfrac{19}{4}$$

we need $F(x)$ so that

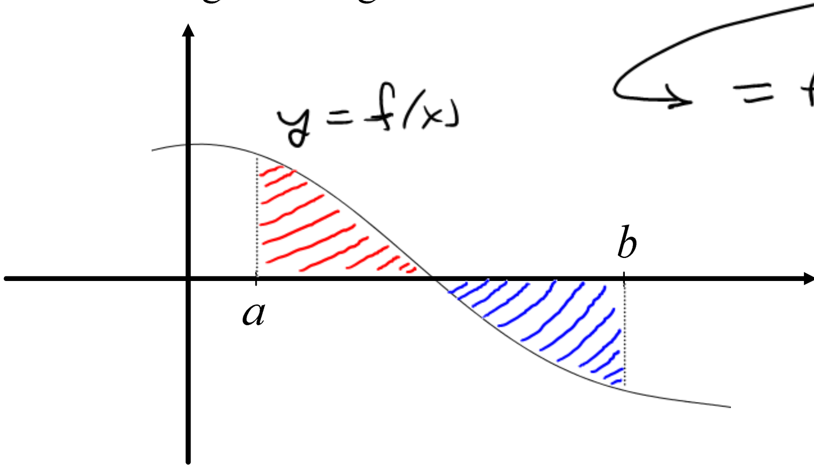
$F'(x) = x^3+1$

$F(x) = \frac{1}{4}x^4 + x$

The **Riemann** Integral of f from a to b: $\int_a^b f(x)dx$

$= \text{Area}(\quad) - \text{Area}(\quad)$

2. What do we get in the general case?



i.e. $\int_a^b f(x)dx = (\text{Area above the x-axis}) - (\text{Area below the x-axis})$

$= \underline{\text{Net Area}}.$

Example: Compute $\int_1^4 \left(x^2 - 3x\right) dx$. Then give a geometric interpretation for this integral.

$$\int_1^4 (x^2-3x)\,dx = \left(\frac{1}{3}x^3 - \frac{3}{2}x^2\right)\Bigg|_1^4$$

$$= \left(\frac{64}{3} - 24\right) - \left(\frac{1}{3} - \frac{3}{2}\right)$$

$$= \frac{63}{3} - 24 + \frac{3}{2}$$

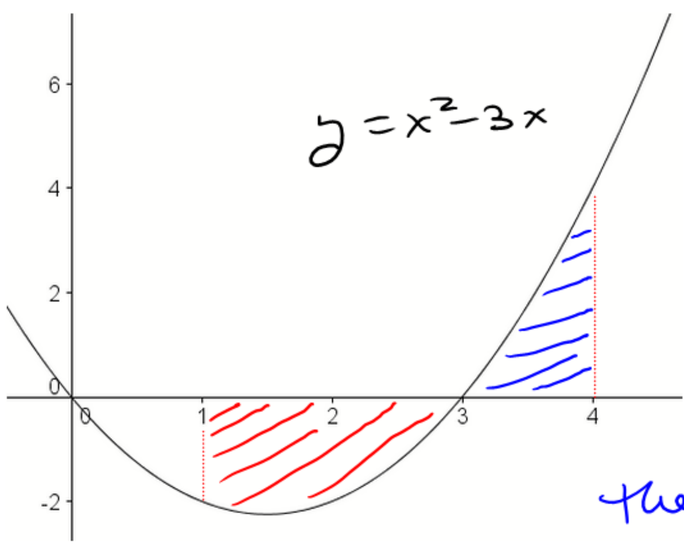
$$= 21 - 24 + \frac{3}{2} = -3 + \frac{3}{2}$$

$$= -\frac{3}{2}.$$

We need $F(x)$ so that

$F'(x) = x^2 - 3x$

$F(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2$



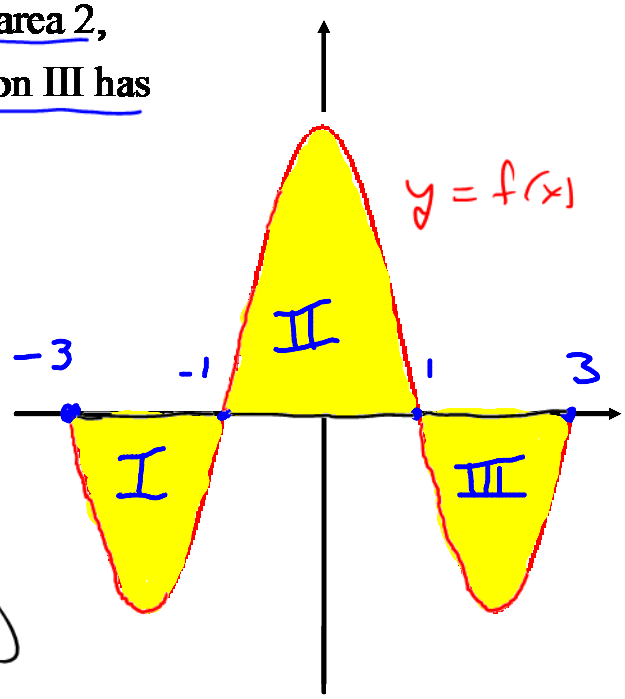
Note that $x^2 - 3x$ becomes negative on a portion of $[1,4]$.

So $\int_1^4 (x^2-3x)\,dx$ is NOT the area bounded between the graph of $y = x^2 - 3x$ and the x-axis on $[1,4]$.

Here $\int_1^4 (x^2-3x)\,dx$ = Net Area = Area(blue) − Area(red)

$= -\frac{3}{2}.$

Example: $y = f(x)$ is graphed on the right for $-3 \le x \le 3$. Region I has area 2, region II has area 3, and region III has area 2. Give $\int_{-3}^{3} f(x)\,dx$



$\int_{-3}^{3} f(x)dx$ = Net Area

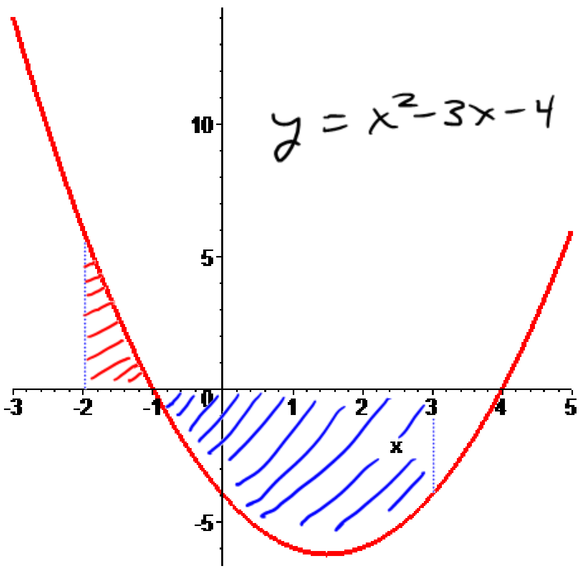
= (Area above)

− (Area below)

= Area(II) − (Area(I) + Area(III))

= 3 − (2 + 2) = −1.

Example: Give the area bounded between the x-axis and the curve $y = x^2 - 3x - 4$ over the interval $[-2, 3]$.



Area = Area( ) + Area( )

Note: Area( ) $= \int_{-2}^{-1} (x^2 - 3x - 4)\,dx$

since $x^2 - 3x - 4 \ge 0$ on $[-2, -1]$.

Also, $x^2 - 3x - 4 \le 0$ on $[-1, 3]$,

So $\int_{-1}^{3} (x^2 - 3x - 4)\,dx = 0 -$ Area( )

$\therefore$ Area( ) $= -\int_{-1}^{3} (x^2 - 3x - 4)\,dx$

Area = Area( ) + Area( ) $= \int_{-2}^{-1} (x^2 - 3x - 4)\,dx + -\int_{-1}^{3} (x^2 - 3x - 4)\,dx$

we need $F(x)$ so that

$F'(x) = x^2 - 3x - 4$

$F(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 4x$

$= \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 - 4x\right)\Big|_{-2}^{-1} - \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 - 4x\right)\Big|_{-1}^{3}$

$$= \left(-\frac{1}{3} - \frac{3}{2} + 4\right) - \left(-\frac{8}{3} - 6 + 8\right) - \left[\left(9 - \frac{27}{2} - 12\right) - \left(-\frac{1}{3} - \frac{3}{2} + 4\right)\right]$$

$$= 9 + \frac{21}{2} + \frac{6}{3} = 21.5$$