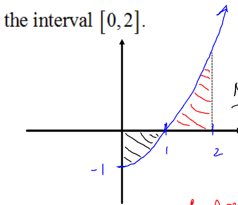
Approximating Riemann Integrals with Riemann Sums

- We are in chapter 5.
- New EMCFs and Homework are posted.
- You have an online quiz due and a practice test due tonight.
- Test 3 ends today.

Popper P24 π

1. $\int_0^{\pi} (2x - \sin(x))\,dx =$

2. Find the area bounded by the graph of $y = x^2 - 1$ and the x-axis over the interval $[0,2]$.



Area = Area(▨) + Area(▨)

Note: $\int_0^1 (x^2-1)\,dx = 0 - \text{Area}(▨)$

$\int_1^2 (x^2-1)\,dx = \text{Area}(▨) - 0$

$\therefore$ Area(▨) $= -\int_0^1 (x^2-1)\,dx$

Area(▨) $= \int_1^2 (x^2-1)\,dx$

$\therefore$ Area $= -\int_0^1 (x^2-1)\,dx + \int_1^2 (x^2-1)\,dx$

$= \dots$

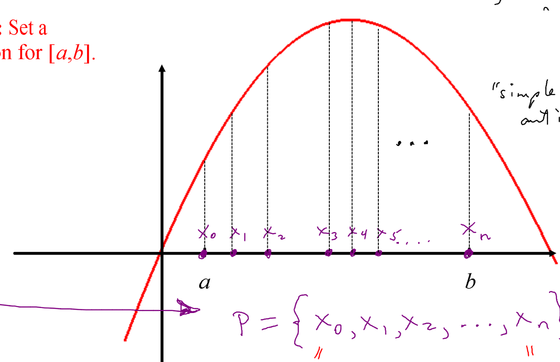
Approximating $\int_a^b f(x)\,dx$ with Riemann Sums

$\int_0^1 \frac{1}{\sqrt{x^3+1}}\,dx$ impossible

← Approx is not needed if we can find an anti-deriv. for $f(x)$.

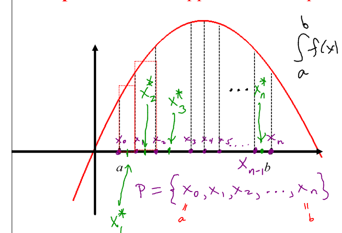
Sad, but true. ^ functions have "simple" anti-deriv.

Step 1: Set a partition for $[a,b]$.



$P = \{x_0, x_1, x_2, \dots, x_n\}$, $x_0 = a$, $x_n = b$

Step 2: Break the approximation up into pieces based upon the partition.



$P = \{x_0, x_1, x_2, \dots, x_n\}$, $x_0 = a$, $x_n = b$

$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \dots + \int_{x_{n-1}}^{x_n} f(x)\,dx$

$\approx f(x_1^*)(x_1 - x_0) + f(x_2^*)(x_2 - x_1) + \dots + f(x_n^*)(x_n - x_{n-1})$

Step 3: Approximation each piece based by selecting a representative function value in each sub-interval.

$= \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$

Riemann Sum $\rightarrow$ $= \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$

Riemann Sum Methods for approximating

$$\int_a^b f(x)dx$$

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$$

$P = \{x_0, x_1, \ldots, x_n\}$ is a partition for $[a,b]$

$$\Delta x_i = x_i - x_{i-1}$$

$$x_i^* \in [x_{i-1}, x_i]$$

Typical Riemann Sums

1. Upper Sum

   $U_f(P)$ Choose x_i^* so that $f(x_i^*) \ge f(x)$ for $x_{i-1} \le x \le x_i$

2. Lower Sum

   $L_f(P)$ Choose x_i^* so that $f(x_i^*) \le f(x)$ for $x_{i-1} \le x \le x_i$.

3. Left Hand Endpoint Method

   $x_i^* = x_{i-1}$

4. Right Hand Endpoint Method

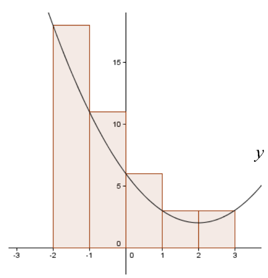
   $x_i^* = x_i$

5. Midpoint Method

   $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$

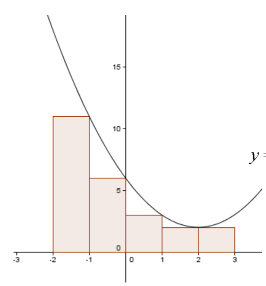
Examples:

Upper Sum for f on [-2,3] with respect to the partition P = {-2,-1,0,1,2,3}



$y = f(x)$

Lower Sum for f on [-2,3] with respect to the partition P = {-2,-1,0,1,2,3}



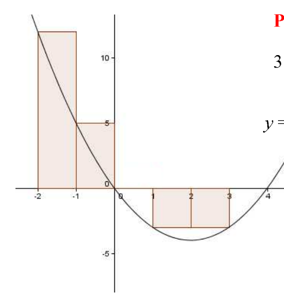
$y = f(x)$

Examples:

Popper P24

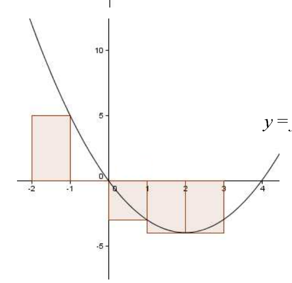
3. 1 + 2 =

Upper Sum for f on [-2,3] with respect to the partition P = {-2,-1,0,1,2,3}



$y = f(x)$

Lower Sum for f on [-2,3] with respect to the partition P = {-2,-1,0,1,2,3}



$y = f(x)$

Regardless of the Choice of Partition P

$$L_f(P) \le \int_a^b f(x)dx \le U_f(P)$$

(and all other Riemann Sums are trapped between these 2)

Theorem: If f is a continuous function on the interval $[a,b]$, then

$$\lim_{|P| \to 0} L_P(f) = \int_a^b f(x)dx$$

and

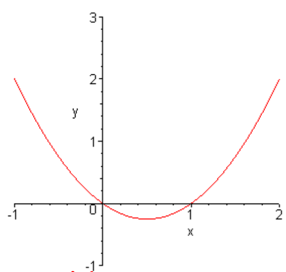
$$\lim_{|P| \to 0} U_P(f) = \int_a^b f(x)dx$$

See the lecture video for more discussion on this point.

Example:

Compute $U_f(P)$ and $L_f(P)$ for the function $f(x) = x^2 - x$ on the interval $[-1, 2]$ with the partition

$$P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}.$$



x	f(x)
-1	2
$\frac{-1}{2}$	$\frac{3}{4}$
0	0
$\frac{1}{2}$	$\frac{-1}{4}$
1	0
$\frac{3}{2}$	$\frac{3}{4}$
2	2

See the lecture video.

Example:

Compute $U_f(P)$ and $L_f(P)$ for the function $f(x) = x^2 - x$ on the interval $[-1, 2]$ with the partition

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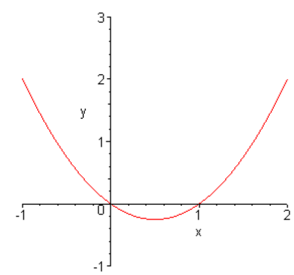


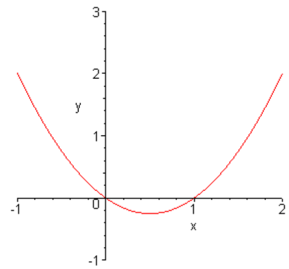
x	f(x)
-1	2
$\frac{-1}{2}$	$\frac{3}{4}$
0	0
$\frac{1}{2}$	$\frac{-1}{4}$
1	0
$\frac{3}{2}$	$\frac{3}{4}$
2	2

See the lecture video.

Example:

Compute Riemann sums using left hand end points and right hand end points for the function $f(x) = x^2 - x$ on the interval $[-1, 2]$ with the partition

$$P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}.$$



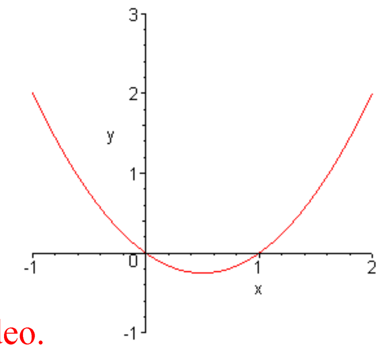
x	$f(x)$
-1	2
$\frac{-1}{2}$	$\frac{3}{4}$
0	0
$\frac{1}{2}$	$\frac{-1}{4}$
1	0
$\frac{3}{2}$	$\frac{3}{4}$
2	2

See the lecture video.

Example:

Compute the Riemann sum using midpoints for the function $f(x) = x^2 - x$ on the interval $[-1,2]$ with respect to the partition

$$P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}.$$



See the lecture video.