Approximating Riemann Integrals with Riemann Sums

- We are in chapter 5.
- New EMCFs and Homework are posted.
- You have an online quiz due and a practice test due tonight.
- Test 3 ends today.

1. \( \int (2x - \sin(x)) \, dx = \)

2. Find the area bounded by the graph of \( y = x^2 - 1 \) and the x-axis over the interval \([0, 2]\).

Step 1: Set a partition for \([a, b]\).

Step 2: Break the approximation up into pieces based upon the partition.

\[ \sum f(x^*) \Delta x \]

Step 3: Approximation each piece based by selecting a representative function value in each sub-interval.

\[ R_{\text{Riemann}} = \sum_{k=1}^{n} f(x^*_k) \Delta x_k \]
Riemann Sum Methods for approximating \( \int_a^b f(x) \, dx \)

\[
\int_a^b f(x) \, dx \approx \sum_{i=1}^{n} f(x_i^*) \Delta x_i
\]

\( P = \{x_0, x_1, \ldots, x_n\} \) is a partition for \([a, b]\)

\( \Delta x_i = x_i - x_{i-1} \)

\( x_i^* \in [x_{i-1}, x_i] \)

**Typical Riemann Sums**

1. Upper Sum
   \( U_f(P) \)
   \( f(x_i^*) \geq f(x) \) for \( x_{i-1} \leq x \leq x_i \)

2. Lower Sum
   \( L_f(P) \)
   \( f(x_i^*) \leq f(x) \) for \( x_{i-1} \leq x \leq x_i \)

3. Left Hand Endpoint Method
   \( x_i^* = x_{i-1} \)

4. Right Hand Endpoint Method
   \( x_i^* = x_i \)

5. Midpoint Method
   \( x_i^* = \frac{1}{2}(x_{i-1} + x_i) \)

**Examples:**

Upper Sum for \( f \) on \([-2, 3]\) with respect to the partition \( P = \{-2, -1, 0, 1, 2, 3\} \)

Lower Sum for \( f \) on \([-2, 3]\) with respect to the partition \( P = \{-2, -1, 0, 1, 2, 3\} \)

**Popper P24**

Upper Sum for \( f \) on \([-2, 3]\) with respect to the partition \( P = \{-2, -1, 0, 1, 2, 3\} \)

Lower Sum for \( f \) on \([-2, 3]\) with respect to the partition \( P = \{-2, -1, 0, 1, 2, 3\} \)
Regardless of the Choice of Partition $P$

$$
L_f(P) \leq \int_a^b f(x) \, dx \leq U_f(P)
$$

(and all other Riemann Sums are trapped between these 2)

**Theorem:** If $f$ is a continuous function on the interval $[a,b]$, then

$$
\lim_{|P| \to 0} L_f(P) = \int_a^b f(x) \, dx
$$

and

$$
\lim_{|P| \to 0} U_f(P) = \int_a^b f(x) \, dx
$$

See the lecture video for more discussion on this point.

**Example:**

Compute $U_f(P)$ and $L_f(P)$ for the function $f(x) = x^2 - x$ on the interval $[-1,2]$ with the partition

$P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$

See the lecture video.

**Example:**

Compute $U_f(P)$ and $L_f(P)$ for the function $f(x) = x^2 - x$ on the interval $[-1,2]$ with the partition

$P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$

See the lecture video.
Example:

Compute Riemann sums using left hand end points and right hand end points for the function \( f(x) = x^2 - x \) on the interval \([-1, 2]\) with the partition
\[
P = \left\{ -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \right\}.
\]

See the lecture video.

Example:

Compute the Riemann sum using midpoints for the function \( f(x) = x^2 - x \) on the interval \([-1, 2]\) with respect to the partition
\[
P = \left\{ -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \right\}.
\]

See the lecture video.