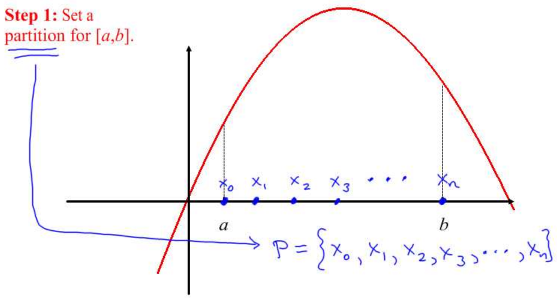
Approximating Riemann Integrals with Riemann Sums

- We are in chapter 5.
- New EMCFs and Homework are posted.
- You have an online quiz due and a practice test due tonight.
- Test 3 ends today.

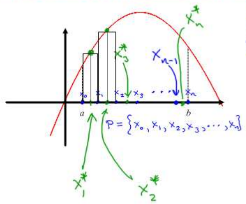
Approximating $\int_a^b f(x)dx$

with Riemann Sums

Step 1: Set a partition for $[a,b]$.



Step 2: Break the approximation up into pieces based upon the partition.



$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

Step 3: Approximation each piece based by selecting a representative function value in each sub-interval.

$$\int_a^b f(x)dx \approx f(x_1^*)\underbrace{(x_1-x_0)}_{\Delta x_1} + f(x_2^*)\underbrace{(x_2-x_1)}_{\Delta x_2} + \cdots + f(x_n^*)\underbrace{(x_n-x_{n-1})}_{\Delta x_n}$$

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Riemann Sum Methods for approximating

Riemann Integral $\rightarrow$ $\int_a^b f(x)dx$

Riemann Sum

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

$P = \{x_0, x_1, \dots, x_n\}$ is a partition for $[a,b]$

$$\Delta x_i = x_i - x_{i-1}$$

$$x_i^* \in [x_{i-1}, x_i]$$

Typical Riemann Sums

1. Upper Sum

   $U_f(P)$ choose x_i^* so that $f(x_i^*) \geq f(x)$ for $x_{i-1} \leq x \leq x_i$

2. Lower Sum

   $L_f(P)$ choose x_i^* so that $f(x_i^*) \leq f(x)$ for $x_{i-1} \leq x \leq x_i$

3. Left Hand Endpoint Method

   choose $x_i^* = x_{i-1}$

4. Right Hand Endpoint Method

   choose $x_i^* = x_i$

5. Midpoint Method

   choose $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$

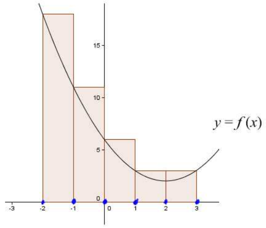
   Only good approximation (of these)

Examples:

Upper Sum for f on [-2,3] with respect to the partition $P = \{-2,-1,0,1,2,3\}$

choose x_i^* so that $f(x_i^*) \geq f(x)$ for $x_{i-1} \leq x \leq x_i$

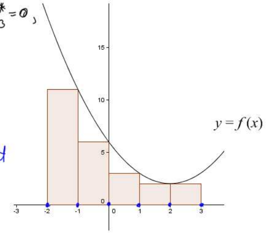
$x_1^* = -2$, $x_2^* = -1$, $x_3^* = 0$, $x_4^* = 1$, $x_5^* = 3$



Lower Sum for f on [-2,3] with respect to the partition $P = \{-2,-1,0,1,2,3\}$

choose x_i^* so that $f(x_i^*) \leq f(x)$ for $x_{i-1} \leq x \leq x_i$

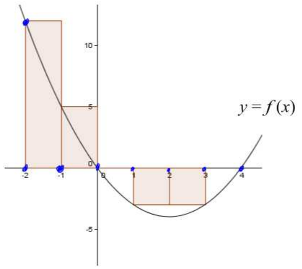
$x_1^* = -1$, $x_2^* = 0$, $x_3^* = 1$, $x_4^* = 2$, $x_5^* = 2$



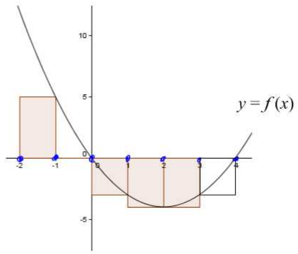
Examples:

Upper Sum for f on [-2,3] with respect to the partition $P = \{-2,-1,0,1,2,3\}$

$x_1^* = -2$, $x_2^* = -1$, $x_3^* = 0$, $x_4^* = 1$, $x_5^* = 3$, $x_6^* = 4$



Lower Sum for f on [-2,3] with respect to the partition $P = \{-2,-1,0,1,2,3\}$



Regardless of the Choice of Partition P

Important

$$L_f(P) \leq \int_a^b f(x)dx \leq U_f(P)$$

(and all other Riemann Sums are trapped between these 2)

Theorem: If f is a continuous function on the interval $[a,b]$, then

$$\lim_{|P|\to 0} L_P(f) = \int_a^b f(x)\,dx$$

and

$$\lim_{|P|\to 0} U_P(f) = \int_a^b f(x)\,dx$$

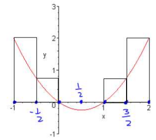
$|P|$ = mesh size of P

$= \max \Delta x_i$

As will the others.

Example:

Compute $U_f(P)$ and $L_f(P)$ for the function $f(x)=x^2-x$ on the interval $[-1,2]$ with the partition $P=\left\{-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2},2\right\}$.



x	$f(x)$
-1	2
$\frac{-1}{2}$	$\frac{3}{4}$
0	0
$\frac{1}{2}$	$\frac{-1}{4}$
1	0
$\frac{3}{2}$	$\frac{3}{4}$
2	2

$$f(-1)\left(\tfrac{1}{2}\right) + f\left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + f(0)\left(\tfrac{1}{2}\right) + f(1)\left(\tfrac{1}{2}\right) + f\left(\tfrac{3}{2}\right)\left(\tfrac{1}{2}\right) + f(2)\left(\tfrac{1}{2}\right)$$

$$= 2\left(\tfrac{1}{2}\right) + \tfrac{3}{4}\left(\tfrac{1}{2}\right) + 0 + 0 + \tfrac{3}{4}\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{2}\right)$$

$$= 1 + \tfrac{3}{8} + \tfrac{3}{8} + 1 = 2.75$$

Note: This is a lousy approx for

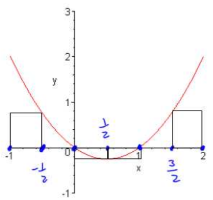
$$\int_{-1}^{2}(x^2-x)\,dx = \left(\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right)\Big|_{-1}^{2}$$

$$= \left(\tfrac{8}{3}-2\right) - \left(-\tfrac{1}{3}-\tfrac{1}{2}\right)$$

$$= \tfrac{2}{3} + \tfrac{1}{3} + \tfrac{1}{2} = 1.5$$

Example:

Compute $U_f(P)$ and $L_f(P)$ for the function $f(x)=x^2-x$ on the interval $[-1,2]$ with the partition $P=\left\{-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2},2\right\}$.



x	$f(x)$
-1	2
$\frac{-1}{2}$	$\frac{3}{4}$
0	0
$\frac{1}{2}$	$\frac{-1}{4}$
1	0
$\frac{3}{2}$	$\frac{3}{4}$
2	2

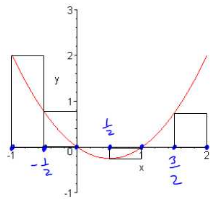
$$f\left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + f(0)\left(\tfrac{1}{2}\right) + f\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + f\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + f(1)\left(\tfrac{1}{2}\right) + f\left(\tfrac{3}{2}\right)\left(\tfrac{1}{2}\right)$$

$$= \tfrac{3}{4}\left(\tfrac{1}{2}\right) + 0 + \left(-\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right) + \left(-\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right) + 0 + \tfrac{3}{4}\left(\tfrac{1}{2}\right)$$

$$= \tfrac{3}{8} - \tfrac{1}{8} - \tfrac{1}{8} + \tfrac{3}{8} = \tfrac{1}{2}.$$

Example:

Compute Riemann sums using left hand end points and right hand end points for the function $f(x)=x^2-x$ on the interval $[-1,2]$ with the partition $P=\left\{-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2},2\right\}$.



x	$f(x)$
-1	2
$\frac{-1}{2}$	$\frac{3}{4}$
0	0
$\frac{1}{2}$	$\frac{-1}{4}$
1	0
$\frac{3}{2}$	$\frac{3}{4}$
2	2

$$f(-1)\tfrac{1}{2} + f\left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + f(0)\tfrac{1}{2} + f\left(\tfrac{1}{2}\right)\tfrac{1}{2} + f(1)\tfrac{1}{2} + f\left(\tfrac{3}{2}\right)\tfrac{1}{2}$$

$$= 2\cdot\tfrac{1}{2} + \tfrac{3}{4}\cdot\tfrac{1}{2} + 0 + \left(-\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right) + 0 + \tfrac{3}{4}\left(\tfrac{1}{2}\right)$$

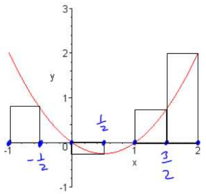
$$= 1 + \tfrac{3}{8} - \tfrac{1}{8} + \tfrac{3}{8} = 1 + \tfrac{5}{8} = \tfrac{13}{8}$$

$$= 1.625$$

Example:

Compute Riemann sums using left hand end points and right hand end points for the function $f(x) = x^2 - x$ on the interval $[-1, 2]$ with the partition

$$P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}.$$



x	$f(x)$
-1	2
$\frac{-1}{2}$	$\frac{3}{4}$
0	0
$\frac{1}{2}$	$\frac{-1}{4}$
1	0
$\frac{3}{2}$	$\frac{3}{4}$
2	2

$$f\left(-\tfrac{1}{2}\right)\cdot\tfrac{1}{2} + f(0)\cdot\tfrac{1}{2} + f\left(\tfrac{1}{2}\right)\cdot\tfrac{1}{2} + f(1)\tfrac{1}{2} + f\left(\tfrac{3}{2}\right)\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2}$$

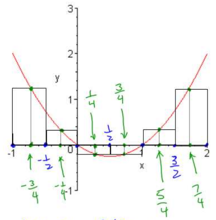
$$= \tfrac{3}{4}\cdot\tfrac{1}{2} + 0 + \left(-\tfrac{1}{4}\right)\cdot\tfrac{1}{2} + 0 + \tfrac{3}{4}\cdot\tfrac{1}{2} + 2\cdot\tfrac{1}{2}$$

$$= \tfrac{3}{8} + -\tfrac{1}{8} + \tfrac{3}{8} + 1 = \tfrac{13}{8} = 1.625$$

Example:

Compute the Riemann sum using midpoints for the function $f(x) = x^2 - x$ on the interval $[-1,2]$ with respect to the partition

$$P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}.$$



$$f\left(-\tfrac{3}{4}\right)\cdot\tfrac{1}{2} + f\left(-\tfrac{1}{4}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{1}{4}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{3}{4}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{5}{4}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{7}{4}\right)\cdot\tfrac{1}{2}$$

$f(x) = x^2 - x$

$$= \left(\tfrac{9}{16} + \cancel{\tfrac{3}{4}}\right)\cdot\tfrac{1}{2} + \left(\tfrac{1}{16} + \cancel{\tfrac{1}{4}}\right)\cdot\tfrac{1}{2} + \left(\tfrac{1}{16} - \cancel{\tfrac{1}{4}}\right)\cdot\tfrac{1}{2} + \left(\tfrac{9}{16} - \cancel{\tfrac{3}{4}}\right)\tfrac{1}{2} + \left(\tfrac{25}{16} - \tfrac{5}{4}\right)\cdot\tfrac{1}{2} + \left(\tfrac{49}{16} - \tfrac{7}{4}\right)\cdot\tfrac{1}{2}$$

$$= \left(\tfrac{94}{16} - \tfrac{12}{4}\right)\cdot\tfrac{1}{2}$$

$$= \left(\tfrac{47}{8} - 3\right)\cdot\tfrac{1}{2} = \tfrac{23}{16} = 1.4375.$$

$$\int_0^1 \sin(x^2)\,dx$$

$$\int_0^1 \frac{1}{\sqrt{x^3+1}}\,dx$$

In these cases, and many others, you must approximate.