Test 4 Online Review
Spring 2012

8 multiple choice - 5 points each
4 written - 15 points each

Starting at 4:00

Topics

Infinite Series:
- Convergence, divergence, absolute convergence, conditional convergence.
- Alternating series and alternating series test.
- Convergence tests for series with nonnegative terms - integral test, comparison test, limit comparison test, ratio test, root test.
- Special series (p-series, geometric series).

L'Hospital's Rule:
- Indeterminant forms.
- Applying the theorem.

Improper Integrals:
- Identification.
- Computation using proper notation.

Taylor Polynomial Approximation:
- Formula for Taylor polynomials.
- Taylor polynomials for simple functions.
- Error estimation and prediction of n to satisfy an error bound.

Practice Questions

$$\sum_{k=2}^{\infty}\left(\frac{e}{2}\right)^k \text{ diverges b/c } \frac{e}{2}>1.$$

Example: Give the values of

$$\sum_{k=0}^{\infty}\frac{1}{2^k},\ \sum_{k=2}^{\infty}\frac{(-1)^k}{2^k},\ \sum_{k=3}^{\infty}\frac{\cos(k\pi)}{2^k},\ \sum_{k=4}^{\infty}\frac{4-2^{k+1}}{3^k},\ \sum_{k=2}^{\infty}\frac{e^k}{2^k}$$

Geometric Series

$$\sum_{k=m}^{\infty} r^k = \begin{cases} r^m \cdot \frac{1}{1-r} & |r|<1 \\ \text{diverges} & |r|\geq 1 \end{cases}$$

$$\sum_{k=0}^{\infty}\frac{1}{2^k} = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = 2$$

Note: $\left|\frac{1}{2}\right|<1$

$$\sum_{k=3}^{\infty}\frac{\cos(k\pi)}{2^k} = \sum_{k=3}^{\infty}\left(-\frac{1}{2}\right)^k = \left(-\frac{1}{2}\right)^3\cdot\frac{1}{1--\frac{1}{2}}$$

$$\sum_{k=2}^{\infty}\frac{(-1)^k}{2^k} = \sum_{k=2}^{\infty}\left(-\frac{1}{2}\right)^k = \left(-\frac{1}{2}\right)^2\cdot\frac{1}{1--\frac{1}{2}} = \frac{1}{4}\cdot\frac{1}{3/2} = \frac{1}{4}\cdot\frac{2}{3} = \frac{1}{6}$$

$\left|-\frac{1}{2}\right|<1$

$$\sum_{k=4}^{\infty}\frac{4-2^{k+1}}{3^k} = \sum_{k=4}^{\infty}\frac{4}{3^k} - \sum_{k=4}^{\infty}\frac{2^{k+1}}{3^k} = 4\sum_{k=4}^{\infty}\left(\frac{1}{3}\right)^k - 2\sum_{k=4}^{\infty}\left(\frac{2}{3}\right)^k$$

$$= 4\cdot\left(\frac{1}{3}\right)^4\cdot\frac{1}{1-\frac{1}{3}} - 2\cdot\left(\frac{2}{3}\right)^4\cdot\frac{1}{1-\frac{2}{3}}$$

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{k=0}^{\infty}\frac{2k}{\sqrt{k^5+2k+1}},\ \sum_{k=2}^{\infty}\frac{(-1)^k k}{k^2+3},\ \sum_{k=3}^{\infty}\frac{2^k}{3k^5},\ \sum_{k=4}^{\infty}\frac{2^{2k+1}}{k!},\ \sum_{k=2}^{\infty}\left(1-\frac{2}{k}\right)^{3k}$$

Converge

LCT with

$\sum \frac{1}{k^{3/2}}$

Alternating Series.

$\sum (-1)^k \cdot \underbrace{\frac{k}{k^2+3}}$

↑ LOOK

1. positive ✓
2. Decreasing ✓
3. Converge to 0. ✓

Diverges

Terms do not go to 0

Conv.

ratio test

diverges.

Terms do not go to zero.

$$\sum_{K=0}^{\infty} \frac{2K}{\sqrt{K^5+2K+1}}$$

Note: $\sqrt{K^5} = K^{5/2}$

Note: $\frac{K}{K^{5/2}} = \frac{1}{K^{3/2}}$

Try LCT

$$\lim_{K\to\infty} \frac{\frac{2K}{\sqrt{K^5+2K+1}}}{\frac{1}{K^{3/2}}} = \lim_{K\to\infty} \frac{2K^{5/2}}{\sqrt{K^5+2K+1}} = 2 < \infty$$

$\therefore$ Since $\sum \frac{1}{K^{3/2}}$ is a convergent p-series, the LCT tells us our series converges.

$$\sum_{K=2}^{\infty} \frac{(-1)^K K}{K^2+3}$$

Note: This is an alternating series.

$$= \sum_{K=2}^{\infty} (-1)^K \cdot \frac{K}{K^2+3}.$$

We need to show:

1. $\frac{K}{K^2+3} > 0$. Obvious since $K \geq 2$.

2. $\frac{K}{K^2+3}$ is eventually decreasing. ✓

$f(x) = \frac{x}{x^2+3}$, $x \geq 2$

$f'(x) = \frac{(x^2+3) - x \cdot 2x}{(x^2+3)^2} = \frac{3-x^2}{(x^2+3)^2} < 0$

$\therefore$ f is decreasing.

3. $\lim_{K\to\infty} \frac{K}{K^2+3} = 0$

$\therefore$ The alternating series test implies our series converges.

$$\sum_{k=4}^{\infty} \frac{2^{2k+1}}{k!} = \sum_{k=4}^{\infty} \frac{2^{2k} \cdot 2}{k!} = 2 \sum_{k=4}^{\infty} \frac{4^k}{k!}$$

Use ratio test.

$$\lim_{k\to\infty} \frac{\frac{4^{k+1}}{(k+1)!}}{\frac{4^k}{k!}} = \lim_{k\to\infty} \frac{\cancel{4^{k+1}}^{\,4} \cdot \cancel{k!}}{\cancel{(k+1)!}_{\,k+1} \cdot \cancel{4^k}} = \lim_{k\to\infty} \frac{4}{k+1}$$

$$= 0 < 1$$

$\therefore$ by the ratio test, our series converges.

Example: Determine whether the series converges or diverges. Show your work.

$$\underbrace{\sum_{k=1}^{\infty}\frac{2\ln(k)}{k^2}}_{\text{converges}},\ \underbrace{\sum_{k=2}^{\infty}\frac{1}{k\ln(k)}}_{\text{Diverges}},\ \underbrace{\sum_{k=3}^{\infty}\frac{1}{k(\ln(k))^2}}_{\text{converges}},\ \underbrace{\sum_{k=4}^{\infty}\frac{\ln(k)}{k}}_{\text{Diverges}}$$

$$\sum_{k=1}^{\infty}\frac{2\ln(k)}{k^2}$$

<u>Note</u>: $\frac{\ln(k)}{k^{1/2}} \to 0$ as $k\to\infty$

$$0 \le 2\frac{\ln(k)}{k^2} = 2\boxed{\frac{\ln(k)}{k^{1/2}}}\cdot\frac{k^{1/2}}{k^2}$$

smaller than 1 for large k.

$$\le 2\frac{1}{k^{3/2}} \quad \text{for large } k.$$

Also $\sum 2\frac{1}{k^{3/2}} = 2\sum\frac{1}{k^{3/2}}$ converges

b/c $\sum\frac{1}{k^{3/2}}$ is a convergent p-series.

$\therefore$ by the C.T., our series converges.

$$\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$$. Note: $f(x)=\frac{1}{x\ln(x)}$ is positive and decreasing for $x\geq 2$.

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx = \lim_{r\to\infty}\int_2^r \frac{1}{x\ln(x)}\,dx$$

improper

$$= \lim_{r\to\infty} \ln|\ln(x)|\Big|_2^r = \lim_{r\to\infty}\Big[\underbrace{\ln(\underline{\ln(r)})}_{\to\infty} - \ln(\ln(2))\Big]$$

$$= \infty$$

$\therefore$ The integral test implies $\sum \frac{1}{k\ln(k)} = \infty$

$\Rightarrow$ the series diverges.

$$\sum_{k=3}^{\infty} \frac{1}{k(\ln(k))^2}$$ $\qquad f(x)=\frac{1}{x(\ln(x))^2}$ $\quad x\geq 3$

then f is positive and decreasing.

$u=\ln(x)$ $\quad du=\frac{1}{x}dx$

$\int u^{-2}du = -\frac{1}{u}$

$$\int_3^{\infty} \frac{1}{x(\ln(x))^2}\,dx = \lim_{r\to\infty}\int_3^r \frac{1}{x(\ln(x))^2}\,dx$$

improper

$$= \lim_{r\to\infty} \frac{-1}{\ln(x)}\Big|_3^r = \lim_{r\to\infty}\left[\frac{-1}{\ln(r)} + \frac{1}{\ln(3)}\right]$$

$$= \frac{1}{\ln(3)} < \infty$$

$\therefore$ the integral test implies our series converges.

$$\sum_{k=4}^{\infty} \frac{\ln(k)}{k}$$

$\ln(k) \geq 1 \iff k \geq e$

<u>Note</u>: $\frac{\ln(k)}{k} \geq \frac{1}{k}$ for $k \geq 3$

<u>AND</u> $\sum \frac{1}{k}$ is a divergent p-series.

$\therefore$ by the C.T., our series diverges.

Example: Determine whether the series converges or diverges. Show your work.

$$\underbrace{\sum_{k=1}^{\infty} k2^{-k}}_{\text{Converges}},\ \underbrace{\sum_{k=2}^{\infty} \frac{3^k}{7k^5+3k^2+27}}_{\text{Diverges}},\ \underbrace{\sum_{k=3}^{\infty} \frac{2k^3+k+1}{k^5+6k^4+1}}_{\text{Converges}},\ \underbrace{\sum_{k=4}^{\infty} \frac{k^2+2}{\sqrt{4k^6+1}}}_{\text{Diverges}}$$

$\sqrt{k^6} = k^3$

$\frac{k^2}{k^3} = \frac{1}{k}$

Converges

$\sum \frac{k}{2^k}$

root or ratio

Diverges

3^k is much larger than $7k^5+3k^2+27$ for large k

$\Rightarrow$ terms do not go to zero.

$\therefore$ series diverges.

Converges

LCT with $\sum \frac{1}{k^2}$

Diverges

LCT with $\sum \frac{1}{k}$

$$\sum_{k=1}^{\infty} k\,2^{-k} = \sum_{k=1}^{\infty} \frac{k}{2^k}$$

$$\lim_{k\to\infty} k^{1/k} = 1$$

root test:

$$\lim_{k\to\infty} \left(\frac{k}{2^k}\right)^{1/k} = \lim_{k\to\infty} \frac{k^{1/k}}{2} = \frac{1}{2} < 1$$

∴ by the root test, our series converges.

ratio test:

$$\lim_{k\to\infty} \frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}} = \lim_{k\to\infty} \frac{(k+1)\cancel{2^k}}{\cancel{2^{k+1}}_{2}\cdot k} = \lim_{k\to\infty} \frac{k+1}{2k}$$

$$= \frac{1}{2} < 1$$

∴ Converges by the ratio test.

$$\sum_{k=4}^{\infty} \frac{k^2+2}{\sqrt{4k^6+1}}$$

Note: $\sqrt{k^6} = k^3$

AND $\frac{k^2}{k^3} = \frac{1}{k}$

$$\lim_{k\to\infty} \frac{\frac{k^2+2}{\sqrt{4k^6+1}}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k(k^2+2)}{\sqrt{4k^6+1}}$$

$$= \lim_{k\to\infty} \frac{k^3+2k}{\sqrt{4k^6+1}} = \frac{1}{2} > 0$$

AND $\sum \frac{1}{k}$ is a divergent p-series.

$\therefore$ by the LCT, our series diverges.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2},\ \sum_{k=2}^{\infty}\frac{(-1)^k}{k},\ \sum_{k=3}^{\infty}\frac{(-1)^k}{\ln(k)},\ \sum_{k=4}^{\infty}\frac{\cos(k\pi)}{k},\ \sum_{k=4}^{\infty}\cos(k\pi)$$

Converges. A.S.T.

Conv. A.S.T.

Conv. A.S.T

Conv. A.S.T.

Diverges. Terms do not go to 0.

$$\sum_{k=3}^{\infty}\frac{(-1)^k}{\ln(k)} = \sum_{k=3}^{\infty}(-1)^k \cdot \frac{1}{\ln(k)}$$

1. $\frac{1}{\ln(k)} > 0$ for $k \geq 3$.

2. $\frac{1}{\ln(k)}$ is decreasing for $k \geq 3$ since $\ln(k)$ is increasing.

3. $\lim_{k\to\infty} \frac{1}{\ln(k)} = 0$

$\therefore$ the A.S.T. $\Rightarrow$ our series converges.

ABS conv $\iff$ $\sum |a_k|$ converges.

COND conv $\iff$ $\begin{cases} 1. \text{ Not abs. conv.} \\ 2. \text{ Converges as is.} \end{cases}$

Diverges $\iff$ You know.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}}_{\text{Cond.}}, \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}}_{\text{abs}}, \underbrace{\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}}_{\text{cond}}, \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3^n}}_{\text{abs}}, \underbrace{\sum_{n=1}^{\infty} \frac{\cos(n\pi)\, n!}{n^n}}_{\text{abs.}}, \underbrace{\sum_{n=1}^{\infty} \frac{\cos(n\pi)\, 2^n}{n!}}_{\text{abs.}}$$

p-series test

root or ratio test

ratio test

ratio test

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$$

1. Check ABS. Conv.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$ ← we did this earlier. Diverges by the integral test.

The series does not converge absolutely.

2. Check cond. conv. (as is, but $\boxed{\text{not abs.}}$)

$$\sum (-1)^n \frac{1}{n \ln(n)}$$ Alternating series!

use the A.S.T. to show convergence (done earlier)

$\therefore$ Our series converges conditionally

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3^n}$$

$$\lim_{n\to\infty} n^{1/n} = 1$$

1. Check ABS conv.

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n}$$

root test

$$\lim_{n\to\infty} \left(\frac{n^2}{3^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{2/n}}{3}$$

$$\boxed{\begin{array}{l} n^{2/n} = n^{\frac{1}{n}\cdot 2} = \left(n^{\frac{1}{n}}\right)^2 \\ a^{bc} = \left(a^b\right)^c \end{array}}$$

$$= \lim_{n\to\infty} \frac{\left(n^{1/n}\right)^2}{3} = \frac{1}{3}$$

$$< 1$$

So, the root test $\Rightarrow$ our series converges absolutely.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=0}^{\infty} \frac{3(-1)^n}{\sqrt{3n^2+2n+1}} \qquad \sum_{n=0}^{\infty} \frac{3n(-1)^n}{\sqrt{3n^2+2n+1}} \qquad \sum_{n=0}^{\infty} \left(\frac{2(-1)^n \arctan n}{3+n^2+n^3} \right)$$

Cond.

ABS fails by LCT with $\sum \frac{1}{n}$.

Cond. by AST.

Diverges.

$\sqrt{n^2} = n$

Terms do not go to zero.

ABS.

Note

$|\arctan(n)| \leq \frac{\pi}{2}$

ABS follows by LCT with $\sum \frac{1}{n^3}$.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(n+1)!}, \sum_{n=1}^{\infty} \frac{(-1)^n n!}{(n+2)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$$

Cond. conv.

ABS fails by LCT with $\sum \frac{1}{n}$.

Cond by A.S.T.

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)(n+1)}$$

ABS by LCT with $\sum \frac{1}{n^2}$

L.H. twice.

not indet. form.

ind. form $\frac{0}{0}$ use L.H.

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{x\to\infty}\frac{\ln(x)}{x},\ \lim_{x\to 0^+}(\cos(x))^{1/x^2},\ \lim_{x\to\infty}\frac{\sin(2x)}{x},\ \lim_{x\to\infty}x\sin\left(\frac{2}{x}\right),\ \lim_{x\to 0}\frac{\sin(3x)-\sin(2x)}{x+\sin(2x)}$$

ind. form

$$\lim_{x\to\infty}\frac{3x^2}{e^x},\ \lim_{x\to 0}\frac{e^{x^2}-1}{3x},\ \lim_{x\to 0^+}\left(\frac{1}{x}\right)^x$$

"0"

" $\lim_{x\to\infty}\frac{\sin(\frac{2}{x})}{\frac{1}{2}\frac{2}{x}}$ " $= 2$

ind. form $\frac{0}{0}$

$\frac{\infty}{\infty}$ Limit is 0.

$\lim_{x\to\infty}\frac{\ln(x)}{x}$ indeterminant $\frac{\infty}{\infty}$

L.H. rule $\lim_{x\to\infty}\frac{1/x}{1} = 0$

$\therefore \lim_{x\to\infty}\frac{\ln(x)}{x} = 0$

$\lim_{x\to 0^+}(\cos(x))^{1/x^2}$ ∞ 1

$(\cos(x))^{\frac{1}{x^2}} = e^{\ln\left[(\cos(x))^{\frac{1}{x^2}}\right]}$

$= e^{\boxed{\frac{\ln(\cos(x))}{x^2}}}$ ← Focus on this.

$\lim_{x\to 0^+}\frac{\ln(\cos(x))}{x^2}$ ind. form. $\frac{0}{0}$.

$$\lim_{x\to 0^+} \frac{\ln(\cos(x))}{x^2}$$ ind. form $\frac{0}{0}$.

L.H. $\lim_{x\to 0^+} \dfrac{\frac{-\sin(x)}{\cos(x)}}{2x} = \lim_{x\to 0^+} -\frac{\sin(x)}{2x}\cdot\frac{1}{\cos(x)}$ $\quad 1 \quad 1$

$= -\frac{1}{2}$

$\therefore \lim_{x\to 0^+} (\cos(x))^{\frac{1}{x^2}} = \lim_{x\to 0^+} e^{\ln((\cos(x))^{1/x^2})} = e^{-\frac{1}{2}}$.

Example: Evaluate the integral using proper notation.

$$\int_0^1 x^{-1/2}\,dx,\ \int_0^2 \frac{2x}{\sqrt{4-x^2}}\,dx,\ \int_{-3}^0 \frac{1}{x+3}\,dx,\ \int_0^\infty \frac{\arctan(x)}{1+x^2}\,dx,\ \int_1^4 x^{-1/2}\,dx$$

improper b/c ∞ limit of integration

not improper.

improper b/c $\frac{1}{\sqrt{x}}$ has a vert. asympt at $x=0$

$\lim\limits_{r\to 0^+} \int_r^1 x^{-\frac{1}{2}}\,dx$

$= \lim\limits_{r\to 0^+} 2\sqrt{x}\Big|_r^1$

$= \lim\limits_{r\to 0^+} \left[2 - 2\sqrt{r}\right]$

$= 2$

improper b/c $\sqrt{4-x^2}=0$ at $x=2$.

$\lim\limits_{tiger\to 2^-} \int_0^{tiger} \frac{2x}{\sqrt{4-x^2}}\,dx$

⋮

improper b/c $x+3=0$ at $x=-3$.

$\lim\limits_{r\to -3^+} \int_r^0 \frac{1}{x+3}\,dx$

Example: Give the 4th degree Taylor polynomial centered at 0 for the given function.

e^x, e^{2x}, e^{-x}, xe^{-x}, $\sin(2x)$, $\cos(3x)$, $x\sin\left(\frac{x}{2}\right)$, $x+2+\ln(1+x)$

e^x: $\quad 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

$f(x) = x + 2 + \ln(1+x)$

$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$

$f(0) = 0 + 2 + \ln(1) = 2$

$f'(x) = 1 + \frac{1}{1+x} \Rightarrow f'(0) = 2$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$

$f^{(4)}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{(4)}(0) = -6$

$= 2 + 2x + \frac{-1}{2}x^2 + \frac{2}{6}x^3 + \frac{-6}{24}x^4$

$= 2 + 2x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

Example: $f(0) = 1$, $f'(0) = -3$, $f''(0) = 0$ and $f'''(0) = 1/4$. Give the 3rd degree Taylor polynomial centered at 0.

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$$

$$= 1 + -3x + \frac{0}{2}x^2 + \frac{1/4}{6}x^3$$

$$= 1 - 3x + \frac{1}{24}x^3.$$

Example: Give the smallest value of n so that the n^{th} degree Taylor polynomial centered at 0 approximates exp(-2) within 10^{-1}.

assume centered at 0.

Example: Give the largest possible error that can occur from approximating e^{-x} on the interval [-1,1] using a Taylor polynomial of degree 5. $\equiv n$

$f(x) = e^{-x}$

$$|\text{error}| \leq \frac{M}{6!}|x|^6 \leq \frac{e}{6!} \cdot 1$$ ← largest possible error is $\frac{e}{6!}$

where x live in $[-1,1]$ and

$$M = \max_{-1 \leq x \leq 1} |f^{(6)}(x)|$$

$$= \max_{-1 \leq x \leq 1} |e^{-x}| = e$$

$f'(x) = -e^{-x}$

$f''(x) = e^{-x}$

$\vdots$

$f^{(6)}(x) = e^{-x}$