Test 2 Review

- Inverse functions
- Logarithmic functions
- Exponential functions
- Logarithmic differentiation
- Exponential growth and decay (word problems)
- Inverse trig functions
- Hyperbolic functions
- Integration by parts
- Integration of powers and products of trig functions

7.1 – 8.3

Addition Problems: See the homework, examples given in the class notes, questions from poppers, questions from EMCFs, questions in online quizzes, review problems and videos posted from the lectures page, and questions given on Friday quizzes.

Also Practice Test 2

Example: Show that $f(x) = x^3 + 4x$ is invertible, and give $(f^{-1})'(5)$.

$f'(x) = 3x^2 + 4 \leftarrow$ always positive

$\therefore f$ is increasing $\Rightarrow f$ is invertible.

Also $(f^{-1})'(5) = \frac{1}{f'(a)}$ where $f(a) = 5$.

$a = 1$

$= \frac{1}{f'(1)} = \frac{1}{7}$.

Example: $f(x) = \frac{x+7}{x+1}$. Give an equation for the tangent line to the graph of $f^{-1}(x)$ at $x = 4$.

Point: $(4, f^{-1}(4)) = (4, 1)$

Solve $\frac{x+7}{x+1} = 4$.

$x = 1$.

slope: $(f^{-1})'(4) = \frac{1}{f'(1)} = \frac{-2}{3}$

$f'(x) = \frac{(x+1)-(x+7)}{(x+1)^2}$

$= \frac{-6}{(x+1)^2}$

$f'(1) = -\frac{3}{2}$

Tangent Line:

$y - 1 = -\frac{2}{3}(x-4)$.

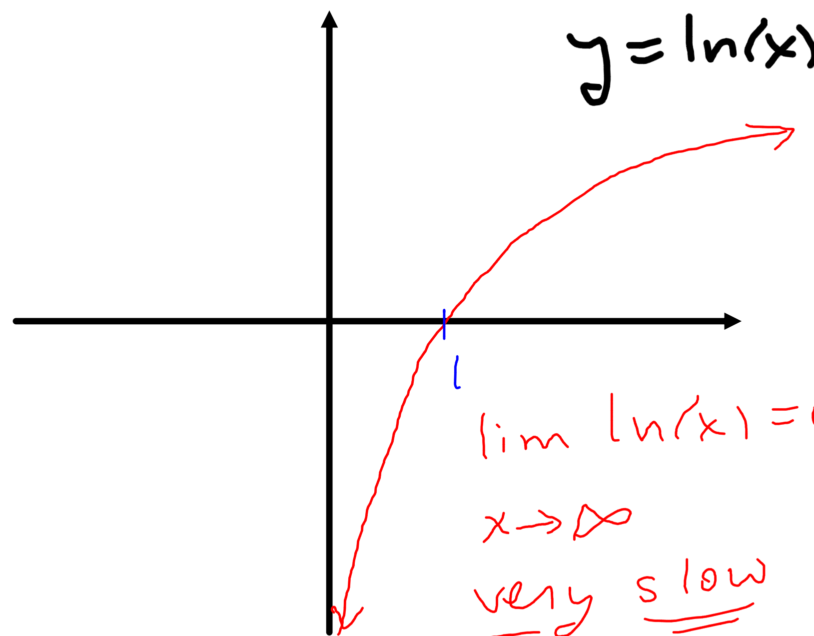
Logarithmic functions (domains, derivatives, etc...): $\log_a(x)$, $\ln(x)$

Domain: $(0, \infty)$ $\quad a > 0$, $a \neq 1$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$



$$\lim_{x\to\infty} \ln(x) = \infty$$

very slow

$$\lim_{x\to 0^+} \ln(x) = -\infty$$

Example: $\dfrac{d}{dx}\left(\log_3\left(x^2+1\right)-\ln\left(\cos\left(x\right)+2\right)\right)=$

$$= \frac{1}{(x^2+1)\ln(3)} \cdot 2x - \frac{1}{\cos(x)+2}(-\sin(x))$$

$$= \frac{2x}{(x^2+1)\ln(3)} + \frac{\sin(x)}{\cos(x)+2}$$

Example: Compute $\displaystyle\int_1^4 \frac{1}{\sqrt{x}\left(\underbrace{2\sqrt{x}+3}\right)}dx.$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$u = 2\sqrt{x} + 3$

$du = \frac{1}{\sqrt{x}}dx$

$$= \int_{\boxed{5}}^{\boxed{7}} \frac{1}{u}\,du$$

$x=4 \Rightarrow u=7$

$x=1 \Rightarrow u=5$

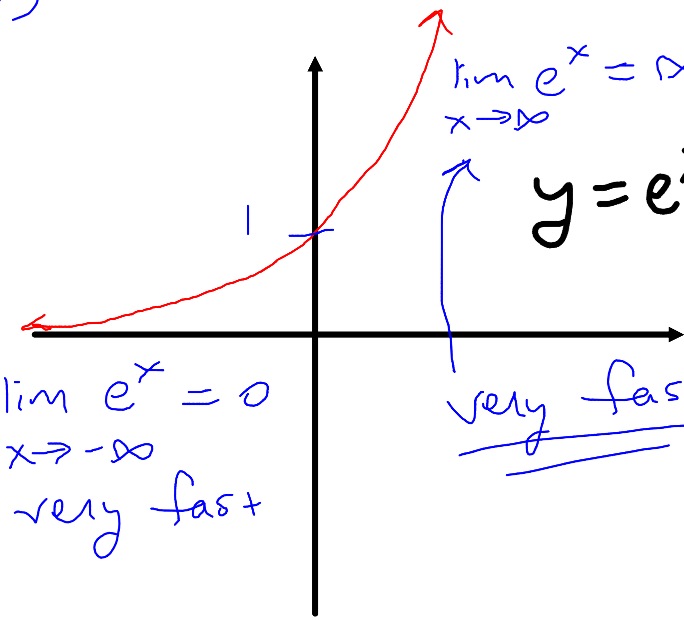
$$= \ln(|u|)\Big|_5^7$$

$$= \ln(7) - \ln(5) = \ln\left(\frac{7}{5}\right).$$

Exponential functions (domain, derivative, etc...): a^x, e^x

$a > 0$, $a \neq 1$.

Domain: $(-\infty, \infty)$



$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} a^x = a^x \ln(a)$$

e^x is the inverse of $\ln(x)$

a^x is the inverse of $\log_a(x)$

$$e^{\ln(x)} = x, \quad \ln(e^x) = x.$$

Example: $f(x) = 4^{x^2} - (\ln(x))^3$. Find $f'(x)$.

$$f'(x) = 4^{x^2} \ln(4) \cdot 2x - 3(\ln(x))^2 \cdot \frac{1}{x}$$

$$= 4^{x^2} \cdot 2x \cdot \ln(4) - \frac{3(\ln(x))^2}{x}$$

Example: Compute $\int \frac{3^{\tan(x)}}{\cos^2(x)}\,dx$. $= \int 3^{\tan(x)} \sec^2(x)\,dx$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$$= \int 3^u\,du$$

$$= \frac{1}{\ln(3)} 3^u + C$$

$$= \frac{1}{\ln(3)} \cdot 3^{\tan(x)} + C$$

Logarithmic differentiation

1. $\left(f(x)\right)^{g(x)}$

2. $f(x)\,g(x)\,h(x)\,G(x)$

$y = \text{expression}$

Goal: Find $\frac{dy}{dx}$

$\ln(y) = \ln(\text{expression})$

use properties of ln.

Diff wrt x.

Solve for $\frac{dy}{dx}$.

Example: $y = (2x+1)^{\sin(x)}$. Find $\dfrac{dy}{dx}$.

$$\ln(y) = \ln\left((2x+1)^{\sin(x)}\right)$$

$$\ln(y) = \sin(x)\ln(2x+1)$$

Diff wrt x

$$\frac{1}{y}\frac{dy}{dx} = \frac{2\sin(x)}{2x+1} + \cos(x)\ln(2x+1)$$

$$\Rightarrow \frac{dy}{dx} = (2x+1)^{\sin(x)}\left[\frac{2\sin(x)}{2x+1} + \cos(x)\ln(2x+1)\right]$$

Exponential growth and decay (word problems)

(A quantity changes at a rate proportional to the amount present)

$$y' = ky$$

$$\Downarrow$$

$$y = \underset{\substack{\uparrow \\ y(0)}}{\underline{\underline{C}}}e^{kt}$$

$\boxed{\text{Note: If our independent variable is } x \text{, then we get } y = Ce^{kx}.}$

ex.

$$u'(t) + 2u(t) = 0, \quad u(0) = 3$$

$$u'(t) = -2u(t)$$

$$u(t) = Ce^{-2t}$$

$$C = 3$$

$$\Rightarrow \quad u(t) = 3 \cdot e^{-2t}$$

Example: A population P of insects increases at a rate proportional to the current population. Suppose there are 10,000 insects initially, and 20,000 insects one week later.
(a) Find an expression for the number of insects $P(t)$ at any time t.
(b) How many insects will there be in 1 year? 2 years?

Example: The half-life of radium-226 is 1620 years. What percentage of a given amount of the radium will remain after 500 years? How long will it take for the original amount to be reduced by 75%?

Example: During the process of inversion, the amount A of raw sugar present decreases at a rate proportional to A. During the first 10 hours, 1000 pounds of raw sugar have been reduced to 800 pounds. How many pounds will remain after 10 more hours of inversion?

A population P of insects increases at a rate proportional to the current population. Suppose there are 10,000 insects initially, and 20,000 insects one week later.
(a) Find an expression for the number of insects $P(t)$ at any time t.
(b) How many insects will there be in 1 year? 2 years?

$P(t) \equiv$ Population at time t in weeks

$P'(t) = k\,P(t)$

initially $\equiv$ "$t=0$"

$\Rightarrow P(t) = C e^{kt}$

$P(0) = 10{,}000$

$P(1) = 20{,}000$

$P(t) = 10000\, e^{kt} = 10000\,(e^k)^t$

$20000 = 10000\, e^k$

$2 = e^k \quad \Rightarrow \quad P(t) = 10000 \cdot 2^t$

a

b

$P(52) = 10000 \cdot 2^{52}$

$P(104) = 10000 \cdot 2^{104}$

$\boxed{\text{The half-life of radium-226 is 1620 years.}}$ What percentage of a given amount of the radium will remain after 500 years? How long will it take for the original amount to be reduced by 75%?

$R(t) \equiv$ amount of radium-226 at time t in years.

Assume: $R'(t) = k R(t)$

$$\Rightarrow \quad R(t) = C e^{kt} = R(0) e^{kt}$$

$$R(1620) = \tfrac{1}{2} R(0)$$

$$\tfrac{1}{2} \cancel{R(0)} = \cancel{R(0)} e^{1620k}$$

$$\ln(1/2) = 1620k$$

$$\boxed{\frac{-\ln(2)}{1620} = k} \quad \Rightarrow \quad R(t) = R(0) e^{\frac{-\ln(2)}{1620} t}$$

$$R(t) = R(0) \cdot 2^{-t/1620}$$

Note: $$\frac{R(500)}{R(0)} = \frac{\cancel{R(0)}\, 2^{-500/1620}}{\cancel{R(0)}} = 2^{-\frac{500}{1620}} \approx .8074$$

"original reduced by 75%" = "25% remains"

$$.25 = \frac{R(t)}{R(0)} = \frac{\cancel{R(0)} \cdot 2^{-t/1620}}{\cancel{R(0)}}$$

$$\ln(.25) = \ln\left(2^{-t/1620}\right) = \frac{-t}{1620} \cdot \ln(2)$$

$$\Rightarrow \quad t = \frac{-1620 \ln(1/4)}{\ln(2)} = \frac{1620 \ln(4)}{\ln(2)}$$

$$= 3240 \text{ years}$$

During the process of inversion, the amount A of raw sugar present decreases at a rate proportional to A. During the first 10 hours, 1000 pounds of raw sugar have been reduced to 800 pounds. How many pounds will remain after 10 more hours of inversion?

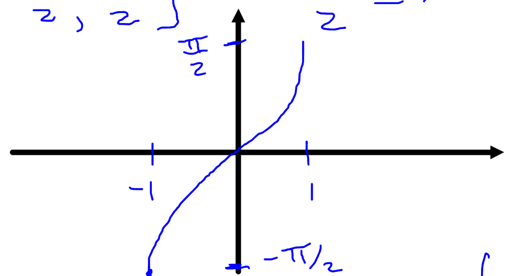
you

Inverse trigonometric functions

$\arcsin(x) \longleftrightarrow$ inverse of $\sin(x)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Domain: $[-1, 1]$. Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

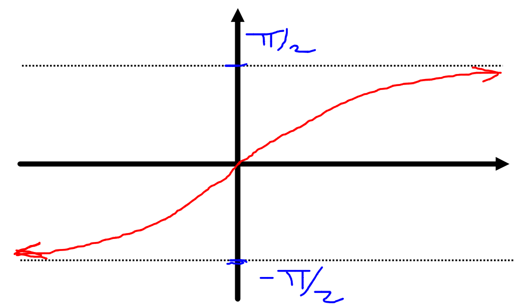


$\arctan(x) \longleftrightarrow$ inverse of $\tan(x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Domain: $(-\infty, \infty)$

Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$



$$\lim_{x\to\infty}\arctan(x) = \frac{\pi}{2}$$

$$\lim_{x\to-\infty}\arctan(x) = -\frac{\pi}{2}$$

$(\sqrt{2}x^2)^2 = 2x^4 \qquad u = \sqrt{2}x^2 \Rightarrow du = 2\sqrt{2}x\,dx$

Example: $\frac{1}{2\sqrt{2}}\int \frac{2\sqrt{2}x}{9+2x^4}dx = \frac{1}{2\sqrt{2}}\int \frac{du}{3^2+u^2} = \frac{1}{6\sqrt{2}}\arctan\left(\frac{u}{3}\right)+C$

$a>0$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$

$$= \frac{1}{6\sqrt{2}}\arctan\left(\frac{\sqrt{2}x^2}{3}\right)+C$$

Example: $\int \frac{x}{\sqrt{9-2x^4}}dx =$ similar. you do it.

$a>0$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right)+C$$

Example: $f(x) = \arcsin(\ln(x))$. Find the domain and give the tangent line at the point where $x = \sqrt{e}$.

Domain: $x > 0$ and $-1 \le \ln(x) \le 1$

$\frac{1}{e} \le x \le e$

$\left[\frac{1}{e}, e\right]$.

T.L.: Point: $(\sqrt{e}, f(\sqrt{e})) = (\sqrt{e}, \arcsin(\tfrac{1}{2})) = (\sqrt{e}, \pi/6)$

Slope: $f'(\sqrt{e}) = \frac{1}{\sqrt{3/4}} \cdot \frac{1}{\sqrt{e}} = \frac{2}{\sqrt{3e}}$

$$f'(x) = \frac{1}{\sqrt{1-(\ln(x))^2}} \cdot \frac{1}{x}$$

$$y - \frac{\pi}{6} = \frac{2}{\sqrt{3e}}(x - \sqrt{e})$$

Example: $f(x) = \sin(\arctan(x))$. Find $f^{-1}\left(\frac{1}{2}\right)$.

Solve $\sin(\arctan(x)) = \frac{1}{2}$.

$\Rightarrow \arctan(x) = \frac{\pi}{6}$

$x = \tan(\pi/6)$

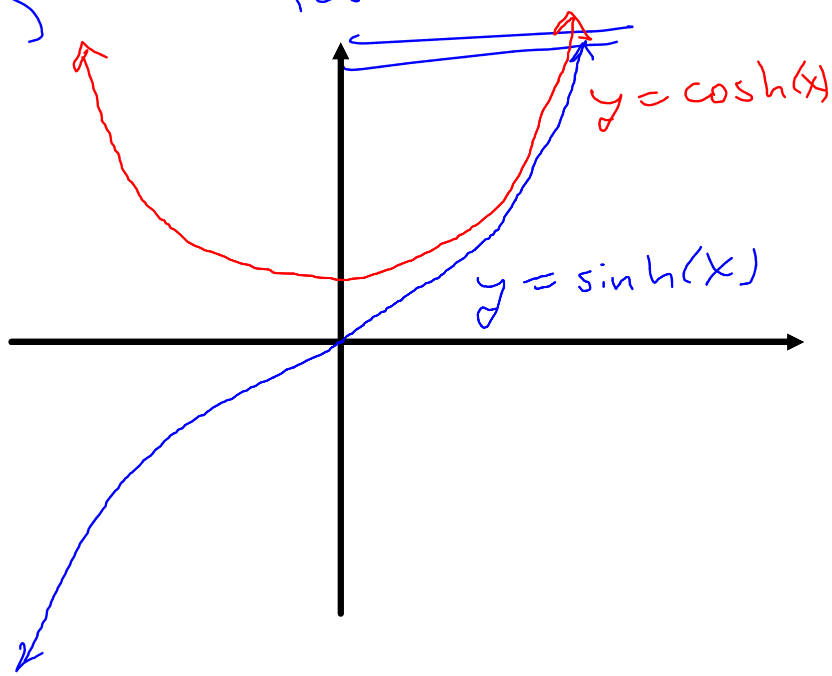
$= \frac{1/2}{\sqrt{3}/2} = 1/\sqrt{3}$.

Hyperbolic functions

$$\cosh(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$$

$$\sinh(x) = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$$

Domain : $(-\infty, \infty)$ for both.



$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

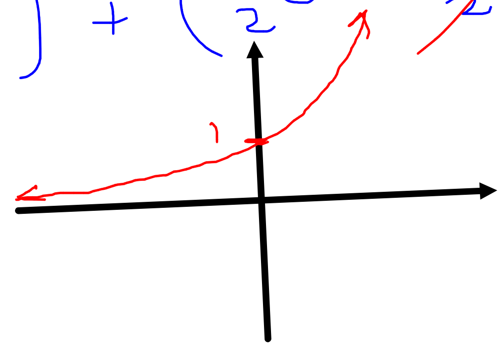
Example: Solve $\cosh(x) - 2\sinh(x) = 0.$

you

Example: Graph $f(x) = \cosh(x) + \sinh(x).$

$$= \left(\frac{1}{2}e^{x} + \cancel{\frac{1}{2}e^{-x}}\right) + \left(\frac{1}{2}e^{x} - \cancel{\frac{1}{2}e^{-x}}\right)$$

$$= e^{x}$$



Example: Differentiate $f(x) = \sinh(x + \ln(x))$.

$$f'(x) = \cosh(x + \ln(x)) \cdot \left(1 + \frac{1}{x}\right)$$

Example: Compute $\int \underline{\cos(x)} \sinh(\underline{\sin(x)}) \underline{dx}$. $= \int \sinh(u)\, du$

$u = \sin(x)$

$du = \cos(x)\, dx$

$= \cosh(u) + C$

$= \cosh(\sin(x)) + C$

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example: Compute $\int_1^e \ln(x)dx$. $= x\ln(x)\Big|_1^e - \int_1^e dx$

$u = \ln(x)$ $\quad du = \frac{1}{x}dx$

$dv = dx$ $\quad v = x$

$= e - 0 - x\Big|_1^e$

$= e - (e-1) = 1$

Example: Compute $\int_0^1 xe^{-x}dx$. $= -xe^{-x}\Big|_0^1 - \int_0^1 -e^{-x}dx$

$u = x$ $\quad du = dx$

$dv = e^{-x}dx$ $\quad v = -e^{-x}$

$= \frac{-1}{e} - 0 - e^{-x}\Big|_0^1$

$= -\frac{1}{e} - \left(\frac{1}{e} - 1\right)$

$= 1 - \frac{2}{e}.$

Integration of powers and products of trig functions

$$\int \sin^m(x)\cos^n(x)\,dx$$

$$\int \sec^m(x)\tan^n(x)\,dx$$

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$1 + \tan^2(x) = \sec^2(x)$$

Example: Compute $\int \sin^4(x)dx$. $= \int (\sin^2(x))^2 dx$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx$$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right) dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{4}\int \cos^2(2x)dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{4}\int \left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right) dx$$

= Finish it.

Example: Compute $\int \sin^2(x)\cos^2(x)dx$. $= \int (\underline{\sin(x)\cos(x)})^2 dx$

similar

$$= \int \frac{1}{4}\sin^2(2x)\,dx$$

$\sin(2x) = 2\underline{\sin(x)\cos(x)}$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos(4x)\right) dx$$

Finish it.

Example: Compute $\int \sin^3(x)\cos^3(x)dx.$ $= \int \sin^3(x)\cos^2(x)\underline{\underline{\cos(x)dx}}$

$$= \int \sin^3(x)\left(1 - \sin^2(x)\right)\cos(x)dx$$

$$= \int \sin^3(x)\cos(x)dx - \int \sin^5(x)\cos(x)dx$$

$$= \frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C$$

Example: Compute $\int \tan^4(x)dx.$ $= \int \tan^2(x)\tan^2(x)dx$

$$= \int \tan^2(x)\left(\sec^2(x) - 1\right)dx$$

$$= \int \tan^2(x)\sec^2(x)dx - \int \tan^2(x)dx$$

$$= \frac{1}{3}\tan^3(x) - \int\left(\sec^2(x) - 1\right)dx$$

$$= \frac{1}{3}\tan^3(x) - \tan(x) + x + C.$$

Example: Compute $\int \tan^3(x)\sec^3(x)dx$.

$$= \int \tan^2(x)\sec^2(x)\,\underline{\sec(x)\tan(x)dx}$$

$$= \int (\sec^2(x)-1)\sec^2(x)\sec(x)\tan(x)dx$$

$$= \int \sec^4(x)\sec(x)\tan(x)dx - \int \sec^2(x)\sec(x)\tan(x)\,dx$$

$$= \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C$$

Example: Compute $\int \tan^2(x)\sec^3(x)dx$.

Parts

$$= \int \underbrace{\tan^2(x)\sec(x)}_{u}\,\underbrace{\sec^2(x)\,dx}_{dv}$$

$u = \tan^2(x)\sec(x)$ $du = ?$

$dv = \sec^2(x)dx \Rightarrow v = ?$

Cyclic

Solve.