Trigonometric Substitution

Section 8.4

Trigonometric Substitution

The following terms can sometimes cause trouble in integration problems:

$a > 0$

$$\sqrt{\underbrace{a^2 - x^2}} \qquad \sqrt{a^2 + x^2} \qquad \sqrt{x^2 - a^2}$$

$x = a\tan(\theta)$ $\qquad$ $x = a\sec(\theta)$

Fortunately, these terms can be collapsed by using trigonometric identifies.

$$\sqrt{1 - x^2} = \sqrt{1 - \sin^2(\theta)} = \sqrt{\cos^2(\theta)}$$

$$= \cos(\theta)$$

$x = \sin(\theta)$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$a > 0$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2(\theta)} = a\sqrt{1 - \sin^2(\theta)}$$

$$= a\cos(\theta)$$

$x = a\sin(\theta)$

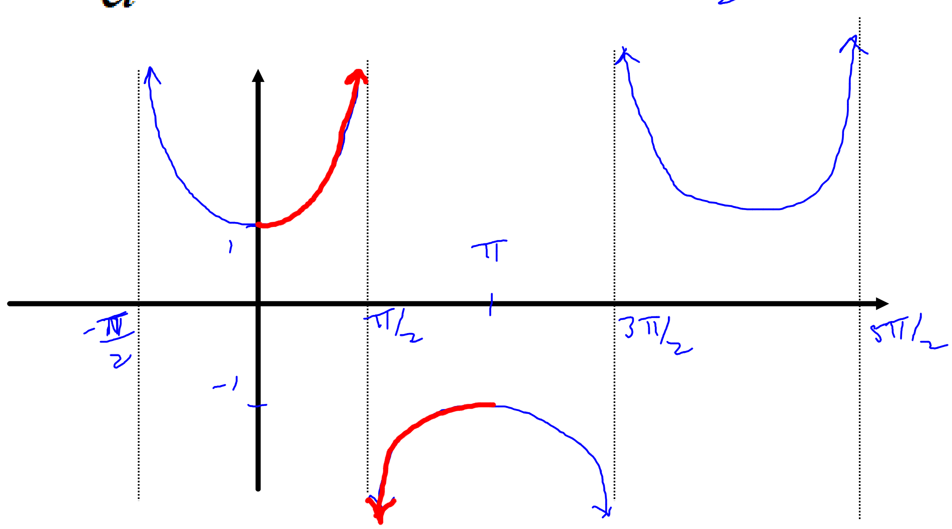
Making Appropriate Substitutions

$$a > 0$$

$$\sqrt{a^2 - x^2} \longleftrightarrow x = a\sin(\theta), \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\sqrt{a^2 + x^2} \longleftrightarrow x = a\tan(\theta), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\sqrt{x^2 - a^2} \longleftrightarrow x = a\sec(\theta), \quad 0 \le \theta < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < \theta \le \pi$$



Examples: What substitution would help with the integration?

$\sqrt{9-x^2}$ $\longleftrightarrow$ $x = 3\sin(\theta)$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\sqrt{16+x^2}$ $\longleftrightarrow$ $x = 4\tan(\theta)$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\sqrt{x^2-4}$ $\longleftrightarrow$ $x = 2\sec(\theta)$

$0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

$\sqrt{2+x^2}$ $\longleftrightarrow$ $x = \sqrt{2}\tan(\theta)$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Examples:

$$\int \frac{x^2}{\sqrt{4+x^2}}\,dx \qquad \int \sqrt{9-x^2}\,dx \qquad \int \sqrt{2-x^2+4x}\,dx$$

$$\int x\sqrt{16+x^2-6x}\,dx \qquad \int \frac{x}{\sqrt{x^2-x-4}}\,dx$$

$$\int \frac{x^2}{\sqrt{4+x^2}}\,dx = \int \frac{4\tan^2(\theta)}{\sqrt{4+4\tan^2(\theta)}}\,2\sec^2(\theta)\,d\theta$$

$x = 2\tan(\theta)$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 2\sec^2(\theta)\,d\theta$

$$= \int \frac{4\tan^2(\theta)\cdot \cancel{2}\sec^2(\theta)\,d\theta}{\cancel{2}\sqrt{1+\tan^2(\theta)}}$$

$$= \int \frac{4\tan^2(\theta)\sec^2(\theta)\,d\theta}{\sqrt{\sec^2(\theta)}}$$

$$= \int \frac{4\tan^2(\theta)\sec^2(\theta)\,d\theta}{\sec(\theta)}$$

$$= 4\int \tan^2(\theta)\sec(\theta)\,d\theta$$

Finish it!

$$\int \sqrt{9-x^2}\,dx = \int \sqrt{9-9\sin^2(\theta)} \cdot 3\cos(\theta)\,d\theta$$

$x = 3\sin(\theta)$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = 3\cos(\theta)\,d\theta$

$$= 9\int \sqrt{1-\sin^2(\theta)} \cdot \cos(\theta)\,d\theta$$

$$= 9\int \sqrt{\cos^2(\theta)}\,\cos(\theta)\,d\theta$$

$$= 9\int \cos^2(\theta)\,d\theta$$

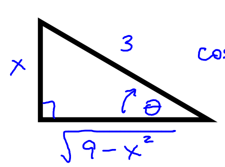
$$= 9\int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)d\theta$$

$\boxed{x = 3\sin(\theta)}$

$\frac{x}{3} = \sin(\theta)$

$\theta = \arcsin\left(\frac{x}{3}\right)$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$



$\cos(\theta) = \frac{\sqrt{9-x^2}}{3}$

$$= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C$$

$$\int \frac{x}{\sqrt{x^2 - x - 4}}\,dx = \int \frac{x}{\sqrt{\left(x-\frac{1}{2}\right)^2 - \frac{17}{4}}}\,dx$$

Key: Completing the square.

$$\underbrace{x^2 - x + \tfrac{1}{4}} \quad -4 - \tfrac{1}{4}$$

$$\left(x - \tfrac{1}{2}\right)^2 - \tfrac{17}{4}$$

$w = x - \frac{1}{2}$

$dw = dx$

$$= \int \frac{w + \frac{1}{2}}{\sqrt{w^2 - \frac{17}{4}}}\,dw$$

$$\boxed{0 \le \theta < \tfrac{\pi}{2} \text{ or } \tfrac{\pi}{2} < \theta \le \pi}$$

$w = \frac{\sqrt{17}}{2}\sec(\theta)$

$dw = \frac{\sqrt{17}}{2}\sec(\theta)\tan(\theta)\,d\theta$

$$= \int \frac{\frac{\sqrt{17}}{2}\sec(\theta) + \frac{1}{2}}{\sqrt{\frac{17}{4}\sec^2(\theta) - \frac{17}{4}}}\ \frac{\sqrt{17}}{2}\sec(\theta)\tan(\theta)\,d\theta$$

$$= \int \frac{\frac{\sqrt{17}}{2}\sec(\theta) + \frac{1}{2}}{\sqrt{\tan^2(\theta)}}\ \sec(\theta)\tan(\theta)\,d\theta$$

The secant substitution leads to some trouble when we deal with the square root.

why?

$\tan(\theta) \ge 0$ for $0 \le \theta < \frac{\pi}{2}$

$\sqrt{\tan^2(\theta)} = \tan(\theta)$

$\tan(\theta) \le 0$ for $\frac{\pi}{2} < \theta \le \pi$

$\sqrt{\tan^2(\theta)} = -\tan(\theta)$

$\sqrt{(-2)^2} \ne -2$

$\quad ||$

$-(-2) = 2$

$$= \int \frac{\frac{\sqrt{17}}{2}\sec(\theta) + \frac{1}{2}}{\sqrt{\tan^2(\theta)}} \sec(\theta)\tan(\theta)\, d\theta$$

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$\tan(\theta) \geq 0$ for $0 \leq \theta < \frac{\pi}{2}$

$$\sqrt{\tan^2(\theta)} = \tan(\theta)$$

$\tan(\theta) \leq 0$ for $\frac{\pi}{2} < \theta \leq \pi$

$$\sqrt{\tan^2(\theta)} = -\tan(\theta)$$

Case 1: $0 \leq \theta < \frac{\pi}{2}$

$$= \int \left(\frac{\sqrt{17}}{2}\sec(\theta) + \frac{1}{2}\right)\sec(\theta)\, d\theta$$

$$= \frac{\sqrt{17}}{2}\int \sec^2(\theta)\, d\theta + \frac{1}{2}\int \sec(\theta)\, d\theta$$

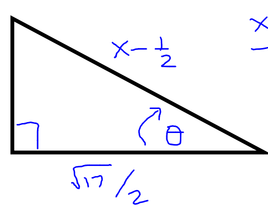
$$= \frac{\sqrt{17}}{2}\tan(\theta) + \frac{1}{2}\ln\left(|\sec(\theta) + \tan(\theta)|\right) + C$$

Case 2: $\frac{\pi}{2} < \theta \leq \pi$

Next page.

Recall: $x - \frac{1}{2} = \frac{\sqrt{17}}{2}\sec(\theta) \geq \frac{\sqrt{17}}{2} \Rightarrow x \geq \frac{1}{2} + \frac{\sqrt{17}}{2}$

$$\sqrt{\left(x-\tfrac{1}{2}\right)^2 - \tfrac{17}{4}}$$



$$\frac{x - \frac{1}{2}}{\sqrt{17}/2} = \sec(\theta)$$

$$\tan(\theta) = \frac{\sqrt{\left(x-\frac{1}{2}\right)^2 - \frac{17}{4}}}{\sqrt{17}/2}$$

$$= \frac{\cancel{\sqrt{17}}}{\cancel{2}} \cdot \frac{\sqrt{\left(x-\frac{1}{2}\right)^2 - \frac{17}{4}}}{\cancel{\sqrt{17}/2}} + \frac{1}{2}\ln\left(\left|\frac{\sqrt{\left(x-\frac{1}{2}\right)^2 - \frac{17}{4}} + x - \frac{1}{2}}{\sqrt{17}/2}\right|\right) + C$$

$$= \sqrt{x^2 - x - 4} + \frac{1}{2}\ln\left(\sqrt{x^2-x-4} + x - \frac{1}{2}\right) + \tilde{C}$$

Case 2: $\frac{\pi}{2} < \theta \le \pi$ $\qquad \tan(\theta) \le 0$

$$\Rightarrow \sqrt{\tan^2(\theta)} = -\tan(\theta)$$

$$= \int \frac{\frac{\sqrt{17}}{2}\sec(\theta) + \frac{1}{2}}{\sqrt{\tan^2(\theta)}} \sec(\theta)\tan(\theta)\, d\theta$$

$$= \int \frac{\frac{\sqrt{17}}{2}\sec(\theta) + \frac{1}{2}}{-\cancel{\tan}(\theta)} \sec(\theta)\cancel{\tan}(\theta)\, d\theta$$

$$= -\int \left(\frac{\sqrt{17}}{2}\sec^2(\theta) + \frac{1}{2}\sec(\theta)\right) d\theta$$

$$= -\frac{\sqrt{17}}{2}\tan(\theta) - \frac{1}{2}\ln\left(\left|\sec(\theta) + \tan(\theta)\right|\right) + C$$

Recall: $\boxed{x - \frac{1}{2} = \frac{\sqrt{17}}{2}\sec(\theta)} \le -\frac{\sqrt{17}}{2}$

and $\quad -\frac{\pi}{2} < \theta \le \pi$

$$\sec(\theta) = \frac{x - \frac{1}{2}}{\sqrt{17}/2} \qquad x \le -\frac{\sqrt{17}}{2} + \frac{1}{2}$$

$$\tan^2(\theta) = \sec^2(\theta) - 1 \Rightarrow \tan(\theta) = \pm\sqrt{\sec^2(\theta) - 1}$$

b/c $-\frac{\pi}{2} < \theta \le \pi$

$$\Rightarrow \tan(\theta) = -\sqrt{\frac{(x - \frac{1}{2})^2}{17/4} - 1}$$

$$= -\frac{2}{\sqrt{17}}\sqrt{(x - \tfrac{1}{2})^2 - \tfrac{17}{4}}$$

$$\tan(\theta) = -\frac{2}{\sqrt{17}}\sqrt{x^2 - x - 4}$$

$$= -\frac{\sqrt{17}}{2}\tan(\theta) - \frac{1}{2}\ln\left(\left|\sec(\theta)+\tan(\theta)\right|\right) + C$$

$$\boxed{\begin{aligned} \tan(\theta) &= \frac{-2}{\sqrt{17}}\sqrt{x^2-x-4} \\ \sec(\theta) &= \frac{x-\frac{1}{2}}{\sqrt{17}/2} \end{aligned}}$$

$$= \sqrt{x^2-x-4} - \frac{1}{2}\ln\left(\left|\frac{x-1/2 - \sqrt{x^2-x-4}}{\sqrt{17}/2}\right|\right) + C$$

$$= \sqrt{x^2-x-4} - \frac{1}{2}\ln\left(\left|x-\frac{1}{2}-\sqrt{x^2-x-4}\right|\right) + \tilde{C}$$

$$= \sqrt{x^2-x-4} + \frac{1}{2}\ln\left(\left|\frac{1}{x-\frac{1}{2}-\sqrt{x^2-x-4}}\right|\right) + \tilde{C}$$

$$= \sqrt{x^2-x-4} + \frac{1}{2}\ln\left(\left|\frac{x-\frac{1}{2}+\sqrt{x^2-x-4}}{\left(x-\frac{1}{2}-\sqrt{x^2-x-4}\right)\left(x-\frac{1}{2}+\sqrt{x^2-x-4}\right)}\right|\right) + \tilde{C}$$

$$= \sqrt{x^2-x-4} + \frac{1}{2}\ln\left(\left|\frac{x-\frac{1}{2}+\sqrt{x^2-x-4}}{\left(x-\frac{1}{2}\right)^2-\left(x^2-x-4\right)}\right|\right) + \tilde{C}$$

Note: $\left(x-\frac{1}{2}\right)^2 - \left(x^2-x-4\right) = \frac{1}{4}+4 = \frac{17}{4}$

$$= \sqrt{x^2-x-4} + \frac{1}{2}\ln\left(\left|\frac{x-\frac{1}{2}+\sqrt{x^2-x-4}}{17/4}\right|\right) + \tilde{C}$$

$$= \sqrt{x^2-x-4} + \frac{1}{2}\ln\left(\left|x-\frac{1}{2}+\sqrt{x^2-x-4}\right|\right) + \tilde{\tilde{C}}$$

$\therefore$

$$\int \frac{x}{\sqrt{x^2-x-4}}\,dx = \begin{cases} \sqrt{x^2-x-4} + \frac{1}{2}\ln\left(\left|\sqrt{x^2-x-4}+x-\frac{1}{2}\right|\right) + \tilde{C} \\ \text{when} \quad x \geq \frac{1}{2} + \frac{\sqrt{17}}{2} \\ \sqrt{x^2-x-4} + \frac{1}{2}\ln\left(\left|x-\frac{1}{2}+\sqrt{x^2-x-4}\right|\right) + \tilde{\tilde{C}} \\ \text{when} \quad x \leq -\frac{\sqrt{17}}{2} + \frac{1}{2} \end{cases}$$

Related, but more complicated...

(The key is completing the square.)

$\sqrt{2-x^2+4x}$ $= \sqrt{2+4 - (\underline{x^2-4x+4})}$

$= \sqrt{6 - (x-2)^2}$

$\sqrt{16+x^2-6x}$ $= \sqrt{16-9 + (\underline{x^2-6x+9})}$

$= \sqrt{7 + (x-3)^2}$

$\sqrt{x^2-x-4}$ we saw this