Partial Fraction Decomposition

Integrating Rational Functions

Examples:

$$\int \frac{x^4+1}{x^3-x^2}dx$$

$$\int \frac{3x+2}{x^3-2x^2+x}dx$$

$$\int \frac{x^2+3x-31}{(x+1)(x^2+4)^2}dx$$

$$\int \frac{x^5}{(x^2+x+1)^2}dx$$

$$\int \frac{x^4-3x^2+1}{x^3-3x^2-4x}dx$$

These are all integrals of rational functions!

Example: Use division to compute $\int \frac{x^3}{x^2+1}dx$

degree 3

degree 2

$$\begin{array}{r} \boxed{x} \\ x^2+1 \overline{\big)\, x^3 + 0x^2 + 0x + 0} \\ -(x^3 + \qquad x) \\ \hline -x \end{array}$$

quotient

remainder

no further division.

$$\therefore \frac{x^3}{x^2+1} = x + \frac{-x}{x^2+1}$$

$$\Rightarrow \int \frac{x^3}{x^2+1}dx = \int \left(x - \frac{x}{x^2+1}\right)dx$$

$$= \frac{1}{2}x^2 - \frac{1}{2}\ln(|x^2+1|) + C$$

In general, we have the following theorem:

If $F(x)$ and $G(x)$ are polynomials and the degree of $F(x)$ is larger than or equal to the degree of $G(x)$, then there are polynomials $q(x)$ and $r(x)$ so that

$$\left\{ \frac{F(x)}{G(x)} = \underline{q(x)} + \frac{r(x)}{G(x)} \right.$$

remainder

where the degree of $r(x)$ is smaller than the degree of $G(x)$.

quotient

q(x) and r(x) come from long division.

Example: Write $\frac{x^5+1}{x^3-x^2-2x}$ in terms of its quotient and remainder.

degree 5

degree 3

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$5 \geq 3$

quotient

$$x^2 + x + 3$$

$$x^3 - x^2 - 2x \,\big)\, x^5 + 0x^4 + 0x^3 + 0x^2 + 0x + 1$$

$$-(x^5 - x^4 - 2x^3)$$

$$x^4 + 2x^3 + 0x^2 + 0x + 1$$

$$-(x^4 - x^3 - 2x^2)$$

$$3x^3 + 2x^2 + 0x + 1$$

$$-(3x^3 - 3x^2 - 6x)$$

$$5x^2 + 6x + 1$$

remainder

$$\therefore \frac{x^5+1}{x^3-x^2-2x} = \underbrace{x^2 + x + 3}_{q(x)} + \frac{\overset{r(x)}{5x^2+6x+1}}{x^3-x^2-2x}$$

The idea behind partial fraction decomposition.

Example: Compute $\int \frac{x^5+1}{x^3-x^2-2x}\,dx$?

Next page.

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2. 4.5

Recall.

$$\frac{x^5+1}{x^3-x^2-2x} = x^2 + x + 3 + \frac{5x^2+6x+1}{x^3-x^2-2x}$$

$$\Rightarrow \int \frac{x^5+1}{x^3-x^2-2x}\,dx = \int \left(x^2 + x + 3 + \frac{5x^2+6x+1}{x^3-x^2-2x}\right)dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x + \int \boxed{\frac{5x^2+6x+1}{x^3-x^2-2x}}\,dx$$

hm...

Idea: $x^3 - x^2 - 2x = x(x^2 - x - 2)$

$= x(x-2)(x+1)$

$$\Rightarrow \frac{5x^2+6x+1}{x^3-x^2-2x} =$$

$$\frac{5x^2+6x+1}{x(x-2)(x+1)} = \frac{(5x+1)\cancel{(x+1)}}{x(x-2)\cancel{(x+1)}}$$

$$= \boxed{\frac{5x+1}{x(x-2)} \overset{?}{=} \frac{A}{x} + \frac{B}{x-2}}$$

The essence of P.F.D.

$$5x + 1 = A(x-2) + Bx$$

$$5x + 1 = Ax - 2A + Bx$$

$-2A = 1 \Rightarrow \boxed{A = -\frac{1}{2}}$

$A + B = 5 \Rightarrow \boxed{B = \frac{11}{2}}$

$$\frac{5x+1}{x(x-2)} = \frac{-1/2}{x} + \frac{11/2}{x-2}$$

$$\int \frac{x^5+1}{x^3-x^2-2x}\,dx =$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x + \int \boxed{\frac{5x^2+6x+1}{x^3-x^2-2x}}\,dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x + \int \left(\frac{-1/2}{x} + \frac{11/2}{x-2}\right)dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x - \frac{1}{2}\ln(|x|) + \frac{11}{2}\ln(|x-2|) + C.$$

Another illustrative example:

Compute $\int \frac{x+1}{x^3-x^2}\,dx$

degree 1

degree 3

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$1 < 3$

$\Rightarrow$ No division.

$$\frac{x+1}{x^3-x^2} = \frac{x+1}{x^2(x-1)}$$

PFD

Maybe

$$\boxed{\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)}}$$

$$x + 1 = Ax(x-1) + B(x-1) + Cx^2$$

$$0x^2 + x + 1 = Ax^2 - Ax + Bx - B + Cx^2$$

$A + C = 0$ $\boxed{C = 2}$

$-A + B = 1$ $\boxed{A = -2}$

$\boxed{B = -1}$

$$\frac{x+1}{x^2(x-1)} = \frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{(x-1)}$$

$$\therefore \int \frac{x+1}{x^3-x^2}\,dx = \int \left(\frac{-2}{x} - \frac{1}{x^2} + \frac{2}{x-1}\right)dx$$

$$= -2\ln(|x|) + \frac{1}{x} + 2\ln(|x-1|) + C$$

Back to partial fraction decomposition...

Steps for computing

degree $F(x) \geq$ degree $G(x)$

$$\int \frac{F(x)}{G(x)}\,dx$$

1. Divide (if necessary) to rewrite the problem in the form

$$\int \left(q(x) + \frac{r(x)}{G(x)} \right) dx$$

where the degree of $r(x)$ is smaller than the degree of $G(x)$.

2. Integrate $q(x)$.

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3. Do a partial fraction decomposition on $\frac{r(x)}{G(x)}$.

How?

1. Factor $G(x)$ into linear and irreducible quadratic factors. $\leftarrow$ no real roots.
2. Rewrite

Always possible.

$$\frac{r(x)}{G(x)} = (\text{terms from linear factors}) + (\text{terms from irred quad})$$

A linear factor of the form $(x-a)^m$ results in terms of the form

$$\frac{b_1}{(x-a)} + \frac{b_2}{(x-a)^2} + \cdots + \frac{b_m}{(x-a)^m}$$

An irreducible quadratic factor of the form $(ax^2+bx+c)^m$ results in terms of the form

$$\frac{d_1x+e_1}{(ax^2+bx+c)} + \frac{d_2x+e_2}{(ax^2+bx+c)^2} + \cdots + \frac{d_mx+e_m}{(ax^2+bx+c)^m}$$

Example: Give the partial fraction decomposition for

$$\frac{2x^2-3x+7}{(x-1)(x+2)(x+3)}$$

$\leftarrow$ degree 2

$\nwarrow$ degree 3

Then integrate the expression.

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$$\frac{2x^2-3x+7}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$A = \frac{1}{2}$, $B = -7$, $C = \frac{17}{2}$

$$2x^2-3x+7 = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)$$

"Killer x's" $x=1$, $x=-2$, $x=-3$

Subst.

$x=1$: $6 = 12A \Rightarrow \boxed{A = \frac{1}{2}}$

$x=-2$: $21 = -3B \Rightarrow \boxed{B = -7}$

$x=-3$: $34 = 4C \Rightarrow \boxed{C = \frac{17}{2}}$

$$\int \frac{2x^2-3x+7}{(x-1)(x+2)(x+3)}\,dx = \int \left(\frac{\frac{1}{2}}{x-1} + \frac{-7}{x+2} + \frac{\frac{17}{2}}{x+3} \right) dx$$

$$= \frac{1}{2}\ln(|x-1|) - 7\ln(|x+2|) + \frac{17}{2}\ln(|x+3|) + C$$

Example: Give the form of the partial fraction decomposition for

$$\frac{2x^2-3x+1}{(x-1)(x+2)^2}$$

How complicated will it be to integrate this expression?

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you

See video.

Example: Compute $\int \frac{1-x^2}{x^3-2x^2+x}dx$

you. see video.

Example: Compute $\int \frac{x^3-2x+1}{x\left(x^2+1\right)^2}dx$

x^3-2x+1 ← degree 3

$x(x^2+1)^2$ ← degree 5

$\boxed{3<5}$

x ↑ Linear factor

x^2+1 ↖ irreducible quadratic

PFD

$$\frac{x^3-2x+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$x^3-2x+1 = A(x^2+1)^2 + (Bx+C)\overbrace{x(x^2+1)}^{(x^3+x)} + \underline{(Dx+E)x}$$

$$x^3-2x+1 = Ax^4+2Ax^2+A + Bx^4+Bx^2+Cx^3+Cx$$
$$+Dx^2+Ex$$

See the video.

Example: Compute $\int \frac{2x^4+3x^2-x+1}{x^2\left(x^2+1\right)^2}dx$