Partial Fraction Decomposition

Integrating Rational Functions

Examples:

$$\int \frac{x^4+1}{x^3-x^2}\,dx$$

$$\int \frac{3x+2}{x^3-2x^2+x}\,dx$$

$$\int \frac{x^2+3x-31}{(x+1)(x^2+4)^2}\,dx$$

$$\int \frac{x^5}{(x^2+x+1)^2}\,dx$$

$$\int \frac{x^4-3x^2+1}{x^3-3x^2-4x}\,dx$$

These are all integrals of rational functions!

Example: Use division to compute $\int \frac{x^3}{x^2+1}\,dx$

degree 3

degree 2

$$
\begin{array}{r}
x \\
x^2+1 \,\big)\, x^3 + 0x^2 + 0x + 0 \\
-\left(x^3 + \qquad x\right) \\
\hline
-x
\end{array}
$$

quotient

remainder

no further division.

$$\therefore \frac{x^3}{x^2+1} = x + \frac{-x}{x^2+1}$$

$$\Rightarrow \int \frac{x^3}{x^2+1}\,dx = \int \left(x - \frac{x}{x^2+1}\right) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{2}\ln\left(|x^2+1|\right) + C$$

In general, we have the following theorem:

If $F(x)$ and $G(x)$ are polynomials and the degree of $F(x)$ is larger than or equal to the degree of $G(x)$, then there are polynomials $q(x)$ and $r(x)$ so that

$$\frac{F(x)}{G(x)} = \underbrace{q(x)}_{\text{quotient}} + \frac{r(x)}{G(x)} \leftarrow \text{remainder}$$

where the degree of $r(x)$ is smaller than the degree of $G(x)$.

q(x) and r(x) come from long division.

Example: Write $\dfrac{x^5+1}{x^3-x^2-2x}$ in terms of its quotient and remainder.

degree 5

degree 3

Popper 08

1. 5

$5 \geq 3$

quotient

$$
\begin{array}{r l}
 & x^2 + x + 3 \\
x^3 - x^2 - 2x \,\big) & x^5 + 0x^4 + 0x^3 + 0x^2 + 0x + 1 \\
 & -(x^5 - x^4 - 2x^3) \\
\hline
 & x^4 + 2x^3 + 0x^2 + 0x + 1 \\
 & -(x^4 - x^3 - 2x^2) \\
\hline
 & 3x^3 + 2x^2 + 0x + 1 \\
 & -(3x^3 - 3x^2 - 6x) \\
\hline
 & 5x^2 + 6x + 1
\end{array}
$$

remainder

$$\therefore \quad \frac{x^5+1}{x^3-x^2-2x} = \underbrace{x^2 + x + 3}_{q(x)} + \frac{\overbrace{5x^2+6x+1}^{r(x)}}{x^3-x^2-2x}$$

The idea behind partial fraction decomposition.

Example: Compute $\int \frac{x^5+1}{x^3-x^2-2x}\,dx$?

Next page.

Popper 08

2. 4.5

Recall.

$$\frac{x^5+1}{x^3-x^2-2x} = x^2+x+3 + \frac{5x^2+6x+1}{x^3-x^2-2x}$$

$$\Rightarrow \int \frac{x^5+1}{x^3-x^2-2x}\,dx = \int\left(x^2+x+3+\frac{5x^2+6x+1}{x^3-x^2-2x}\right)dx$$

$$= \frac{1}{3}x^3+\frac{1}{2}x^2+3x+\int \boxed{\frac{5x^2+6x+1}{x^3-x^2-2x}}\,dx$$

hm...

Idea:

$$x^3-x^2-2x = x(x^2-x-2)$$
$$= x(x-2)(x+1)$$

$$\Rightarrow \frac{5x^2+6x+1}{x^3-x^2-2x} =$$

$$\frac{5x^2+6x+1}{x(x-2)(x+1)} = \frac{(5x+1)\cancel{(x+1)}}{x(x-2)\cancel{(x+1)}}$$

$$= \boxed{\frac{5x+1}{x(x-2)} \overset{?}{=} \frac{A}{x}+\frac{B}{x-2}}$$

The essence of P.F.D.

$$5x+1 = A(x-2)+Bx$$

$$5x+1 = Ax-2A+Bx$$

$$-2A = 1 \Rightarrow \boxed{A=-\frac{1}{2}}$$

$$A+B=5 \Rightarrow \boxed{B=\frac{11}{2}}$$

$$\frac{5x+1}{x(x-2)} = \frac{-1/2}{x}+\frac{11/2}{x-2}$$

$$\int \frac{x^5+1}{x^3-x^2-2x}\,dx =$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x + \int \boxed{\frac{5x^2+6x+1}{x^3-x^2-2x}}\,dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x + \int \left(\frac{-1/2}{x} + \frac{11/2}{x-2} \right) dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x - \frac{1}{2}\ln(|x|) + \frac{11}{2}\ln(|x-2|) + C.$$

Another illustrative example:

degree 1

$$\text{Compute } \int \frac{x+1}{x^3 - x^2}\, dx$$

degree 3

Popper 08

3. 23

$1 < 3$

$\Rightarrow$ No division.

$$\frac{x+1}{x^3-x^2} = \frac{x+1}{x^2(x-1)}$$

PFD

maybe

$$\boxed{\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)}}$$

$$x+1 = Ax(x-1) + B(x-1) + Cx^2$$

$$0x^2 + x + 1 = Ax^2 - Ax + Bx - B + Cx^2$$

$A + C = 0$ $\boxed{C = 2}$

$-A + B = 1$ $\boxed{A = -2}$

$\boxed{B = -1}$

$$\frac{x+1}{x^2(x-1)} = \frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{(x-1)}$$

$$\therefore \int \frac{x+1}{x^3-x^2}\, dx = \int \left(\frac{-2}{x} - \frac{1}{x^2} + \frac{2}{x-1} \right) dx$$

$$= -2\ln(|x|) + \frac{1}{x} + 2\ln(|x-1|) + C$$

Back to partial fraction decomposition...

degree $F(x) \geq$ degree $G(x)$

Steps for computing

$$\int \frac{F(x)}{G(x)}dx$$

1. Divide (if necessary) to rewrite the problem in the form

$$\int\left(q(x) + \frac{r(x)}{G(x)}\right)dx$$

where the degree of $r(x)$ is smaller than the degree of $G(x)$.

2. Integrate $q(x)$.

3. Do a partial fraction decomposition on $\frac{r(x)}{G(x)}$.

How? $\leftarrow$ No real roots.

1. Factor $G(x)$ into linear and irreducible quadratic factors.
2. Rewrite

Always possible.

$$\frac{r(x)}{G(x)} = (\text{terms from linear factors}) + (\text{terms from irred quad})$$

A linear factor of the form $(x-a)^m$ results in terms of the form

$$\frac{b_1}{(x-a)} + \frac{b_2}{(x-a)^2} + \cdots + \frac{b_m}{(x-a)^m}$$

An irreducible quadratic factor of the form $(ax^2+bx+c)^m$ results in terms of the form

$$\frac{d_1x+e_1}{(ax^2+bx+c)} + \frac{d_2x+e_2}{(ax^2+bx+c)^2} + \cdots + \frac{d_mx+e_m}{(ax^2+bx+c)^m}$$

Example: Give the partial fraction decomposition for

$$\frac{2x^2 - 3x + 7}{(x-1)(x+2)(x+3)}$$

degree 2

degree 3

Then integrate the expression.

Popper 08

4. 43

$$\frac{2x^2-3x+7}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$A = \frac{1}{2}$, $B = -7$, $C = \frac{17}{2}$

$$2x^2 - 3x + 7 = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)$$

"Killer x's" $x=1$, $x=-2$, $x=-3$

Subst.

$x=1$: $6 = 12A \Rightarrow \boxed{A = \frac{1}{2}}$

$x=-2$: $21 = -3B \Rightarrow \boxed{B = -7}$

$x=-3$: $34 = 4C \Rightarrow \boxed{C = \frac{17}{2}}$

$$\int \frac{2x^2-3x+7}{(x-1)(x+2)(x+3)}\,dx = \int\left(\frac{1/2}{x-1} + \frac{-7}{x+2} + \frac{17/2}{x+3}\right)dx$$

$$= \frac{1}{2}\ln(|x-1|) - 7\ln(|x+2|) + \frac{17}{2}\ln(|x+3|) + C$$

Example: Give the form of the partial fraction decomposition for

$$\frac{2x^2-3x+1}{(x-1)(x+2)^2}$$

How complicated will it be to integrate this expression?

you

Popper 08

5. 65

See video.

Example: Compute $\int \frac{1-x^2}{x^3-2x^2+x}dx$

you. see video.

Example: Compute $\int \frac{x^3 - 2x + 1}{x\left(x^2+1\right)^2}\,dx$

degree 3

degree 5

$\boxed{3 < 5}$

linear factor

irreducible quadratic

PFD

$$\frac{x^3-2x+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$x^3-2x+1 = A(x^2+1)^2 + (Bx+C)\overbrace{x(x^2+1)}^{(x^3+x)} + (Dx+E)x$$

$$x^3 - 2x + 1 = Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx$$

$$+ Dx^2 + Ex$$

See the video.

Example: Compute $\int \frac{2x^4 + 3x^2 - x + 1}{x^2\left(x^2+1\right)^2}\,dx$