Partial Fraction Decomposition

Integrating Rational Functions

Examples:

$$\int \frac{x^4+1}{x^3-x^2}\,dx$$

$$\int \frac{3x+2}{x^3-2x^2+x}\,dx$$

$$\int \frac{x^2+3x-31}{(x+1)(x^2+4)^2}\,dx$$

$$\int \frac{x^5}{(x^2+x+1)^2}\,dx$$

$$\int \frac{x^4-3x^2+1}{x^3-3x^2-4x}\,dx$$

These are all integrals of rational functions!

Example: Use division to compute $\int \frac{x^3}{x^2+1}\,dx$ degree $= 3$, degree $= 2$

$3 \geq 2$

$$\begin{array}{r} x \\ x^2+1 \overline{)\, x^3 + 0x^2 + 0x + 0} \\ -\left(x^3 \quad + x\right) \\ \hline -x \end{array}$$

$$\therefore \boxed{\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}}$$

$$\Rightarrow \int \frac{x^3}{x^2+1}\,dx = \int \left(x - \frac{x}{x^2+1}\right) dx$$

$$= \frac{x^2}{2} - \frac{1}{2}\ln\left(|x^2+1|\right) + C$$

In general, we have the following theorem:

If $F(x)$ and $G(x)$ are polynomials and the degree of $F(x)$ is larger than or equal to the degree of $G(x)$, then there are polynomials $q(x)$ and $r(x)$ so that

$$\frac{F(x)}{G(x)} = q(x) + \frac{r(x)}{G(x)} \leftarrow \text{remainder}$$

where the degree of $r(x)$ is smaller than the degree of $G(x)$.

quotient

Use long division to find $q(x)$ and $r(x)$.

Example: Write $\dfrac{x^5+1}{x^3-x^2-2x}$ in terms of its quotient and remainder.

← degree 5

← degree 3

$5 \geq 3$

long division

$q(x)$

$$
\begin{array}{r l}
 & x^2 + x + 3 \\
x^3 - x^2 - 2x \,\big) & x^5 + 0x^4 + 0x^3 + 0x^2 + 0x + 5 \\
 & -(x^5 - x^4 - 2x^3) \\
\hline
 & x^4 + 2x^3 + 0x^2 + 0x + 5 \\
 & -(x^4 - x^3 - 2x^2) \\
\hline
 & 3x^3 + 2x^2 + 0x + 5 \\
 & -(3x^3 - 3x^2 - 6x) \\
\hline
 & 5x^2 + 6x + 5
\end{array}
$$

degree $= 3$

degree $= 2$

← $r(x)$

$$\therefore \frac{x^5+1}{x^3-x^2-2x} = x^2 + x + 3 + \frac{5x^2+6x+5}{x^3-x^2-2x}$$

The idea behind partial fraction decomposition.

Example: Compute $\int \frac{x^5+1}{x^3-x^2-2x}\,dx$?

$$\frac{x^5+1}{x^3-x^2-2x} = x^2+x+3 + \frac{5x^2+6x+5}{x^3-x^2-2x}$$

$$\int \frac{x^5+1}{x^3-x^2-2x}\,dx = \int \Big(\underbrace{x^2+x+3}_{\text{easy}} + \underbrace{\frac{5x^2+6x+5}{x^3-x^2-2x}}_{\text{Hm...}} \Big)\,dx$$

Note: $\frac{5x^2+6x+5}{x^3-x^2-2x} = \frac{5x^2+6x+5}{x(x-2)(x+1)}$

I wonder if the expression on the right came from finding a common denominator? It could have...

Note: $\dfrac{5x^2+6x+5}{x^3-x^2-2x} = \dfrac{5x^2+6x+5}{x(x-2)(x+1)}$

I wonder if the expression on the right came from finding a common denominator? It could have...

If so, maybe

$$\frac{5x^2+6x+5}{x(x-2)(x+1)} = \frac{A^{=-5/2}}{x} + \frac{B^{=37/6}}{x-2} + \frac{C^{=4/3}}{x+1}$$

This is the essence of partial fraction decomposition.

Multiply both sides by $x(x-2)(x+1)$.

$$5x^2+6x+5 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

"Killer x's" $\underline{x=0}$, $\underline{x=2}$, $\underline{x=-1}$

Substitute $\underline{x=0}$:

$5 = -2A \Rightarrow A = -\frac{5}{2}$

$\underline{x=2}$:

$37 = 6B \Rightarrow B = \frac{37}{6}$

$\underline{x=-1}$:

$4 = 3C \Rightarrow C = \frac{4}{3}$

$$\int \frac{x^5+1}{x^3-x^2-2x}\,dx = \int \left(x^2 + x + 3 + \frac{5x^2+6x+5}{x^3-x^2-2x} \right) dx$$

$$\boxed{\frac{5x^2+6x+5}{x(x-2)(x+1)} = \frac{A^{=-5/2}}{x} + \frac{B^{=37/6}}{x-2} + \frac{C^{=4/3}}{x+1}}$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 3x + \int \left(\frac{-5/2}{x} + \frac{37/6}{x-2} + \frac{4/3}{x+1} \right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 3x - \frac{5}{2}\ln(|x|) + \frac{37}{6}\ln(|x-2|) + \frac{4}{3}\ln(|x+1|) + C$$

Another illustrative example:

$$\text{Compute } \int \frac{x+1}{x^3 - x^2}\,dx$$

degree = 1

degree = 3

$1 < 3$

$\Rightarrow$ No division.

Note:

$$\frac{x+1}{x^3 - x^2} = \frac{x+1}{x^2(x-1)} \stackrel{?}{=} \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$A = -2$, $B = -1$, $C = 2$

multiply both sides by $x^2(x-1)$.

$$x+1 = Ax(x-1) + B(x-1) + Cx^2$$

"Grocery shopping"

List

$$x+1 = Ax^2 - Ax + Bx - B + Cx^2$$

No x^2 terms in the list!

$\therefore$ $A + C = 0$ $\Rightarrow C = 2$

1 x term in the list!

$-A + B = 1$ $\rightarrow$ $A = -2$

1 constant term.

$-B = 1 \Rightarrow B = -1$

Compute $\int \frac{x+1}{x^3-x^2}\,dx$ $= \int \frac{x+1}{x^2(x-1)}\,dx$

$$= \int \left(\frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{x-1} \right) dx$$

$$= -2\ln(|x|) + \frac{1}{x} + 2\ln(|x-1|) + C$$

Back to partial fraction decomposition...

degree $F(x) \geq$ degree $G(x)$

Steps for computing

$$\int \frac{F(x)}{G(x)}\,dx$$

polynomial

polynomial

1. Divide (if necessary) to rewrite the problem in the form

$$\int \left(q(x) + \frac{r(x)}{G(x)} \right) dx$$

where the degree of $r(x)$ is smaller than the degree of $G(x)$.

2. Integrate $q(x)$.

3. Do a partial fraction decomposition on $\frac{r(x)}{G(x)}$.

no real roots

How?

1. Factor $G(x)$ into linear and irreducible quadratic factors.
2. Rewrite

Amazing Fact: Every polynomial can be factored into linear and irreducible quadratic factors.

$$\frac{r(x)}{G(x)} = (\text{terms from linear factors}) + (\text{terms from irred quad})$$

A linear factor of the form $(x-a)^m$ results in terms of the form

$$\frac{b_1}{(x-a)} + \frac{b_2}{(x-a)^2} + \cdots + \frac{b_m}{(x-a)^m}$$

$b^2-4ac<0$

An irreducible quadratic factor of the form $(ax^2+bx+c)^m$ results in terms of the form

$$\frac{d_1x+e_1}{(ax^2+bx+c)} + \frac{d_2x+e_2}{(ax^2+bx+c)^2} + \cdots + \frac{d_mx+e_m}{(ax^2+bx+c)^m}$$

Example: Give the partial fraction decomposition for

$$\frac{2x^2 - 3x + 7}{(x-1)(x+2)(x+3)}$$

Then integrate the expression.

$$\frac{2x^2 - 3x + 7}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$A = 1/2$, $B = -7$, $C = 17/2$

$$2x^2 - 3x + 7 = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)$$

"killer x's": $x = 1$, $x = -2$, $x = -3$

substitute: $x = 1$: $6 = 12A \Rightarrow A = \frac{1}{2}$.

$x = -2$: $21 = -3B \Rightarrow B = -7$

$x = -3$: $34 = 4C \Rightarrow C = \frac{17}{2}$

$\therefore$ the partial fraction decomposition is

$$\frac{2x^2 - 3x + 7}{(x-1)(x+2)(x+3)} = \frac{1/2}{x-1} + \frac{-7}{x+2} + \frac{17/2}{x+3}$$

$$\int \frac{2x^2 - 3x + 7}{(x-1)(x+2)(x+3)}\,dx = \int \left(\frac{1/2}{x-1} + \frac{-7}{x+2} + \frac{17/2}{x+3} \right) dx$$

$$= \frac{1}{2}\ln(|x-1|) - 7\ln(|x+2|) + \frac{17}{2}\ln(|x+3|) + C$$

Example: Give the form of the partial fraction decomposition for

$$\frac{2x^2-3x+1}{(x-1)(x+2)^2}$$

$\leftarrow$ degree $= 2$

$\leftarrow$ degree $= 3$

How complicated will it be to integrate this expression?

$2 < 3$

$\Rightarrow$ no division

$$\boxed{\frac{2x^2-3x+1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}}$$

This is easy to integrate once we have A, B, C.

Note: $\int\left(\frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\right)dx$

$$= A\ln(|x-1|) + B\ln(|x+2|) - \frac{C}{x+2} + C_1$$

Example: Compute $\int \frac{1-x^2}{x^3-2x^2+x}\,dx$

degree 2

degree 3

PFD

$$\frac{1-x^2}{x^3-2x^2+x} = \frac{1-x^2}{x(x^2-2x+1)} = \frac{1-x^2}{x(x+1)^2}$$

$$\frac{(1-x)(1+x)}{x(x+1)^2} = \frac{1-x}{x(x+1)}$$

$$\frac{1-x}{x(x+1)} = \frac{A}{x} + \frac{C}{x+1}$$

$$1-x = A(x+1) + Cx$$

$$1-x = Ax + A + Cx$$

$A=1$ $\quad A+C=-1 \Rightarrow C=-2$

$\therefore$

$$\int \frac{1-x^2}{x^3-2x^2+x}\,dx = \int \frac{1-x}{x(x+1)}\,dx = \int\left(\frac{1}{x} - \frac{2}{x+1}\right)dx$$

$$= \ln(|x|) - 2\ln(|x+1|) + C$$

Example: Compute $\int \frac{x^3 - 2x + 1}{x(x^2+1)^2}\,dx$

$\leftarrow$ degree $= 3$

$\leftarrow$ degree $= 5$

$3 < 5 \Rightarrow$ No division.

PFD

$$\frac{x^3 - 2x + 1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$A = 1$, $B = -1$, $C = 1$, $D = -1$, $E = -3$

x: linear

x^2+1: irreducible quadratic

$$x^3 - 2x + 1 = A(x^2+1)^2 + (Bx+C)\underbrace{x(x^2+1)}_{(x^3+x)} + (Dx+E)x$$

$x = 0$: $\boxed{1 = A}$

$$x^3 - 2x + 1 = x^4 + 2x^2 + 1 + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$$

No x^4 on LHS: $0 = 1 + B \Rightarrow \boxed{B = -1}$

1 x^3 on LHS: $\boxed{1 = C}$

0 x^2 on LHS: $0 = 2 + B + D \Rightarrow \boxed{D = -1}$

-2 x on LHS: $-2 = C + E \Rightarrow \boxed{E = -3}$

Compute $\int \frac{x^3 - 2x + 1}{x\left(x^2+1\right)^2}\,dx = \int \left(\frac{1}{x} + \frac{-x+1}{x^2+1} + \frac{-x-3}{\left(x^2+1\right)^2} \right) dx$

$$= \ln(|x|) - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx - \frac{1}{2}\int \frac{2x}{\left(x^2+1\right)^2}\,dx - 3\int \frac{1}{\left(x^2+1\right)^2}\,dx$$

$$= \ln(|x|) - \frac{1}{2}\ln\left(|x^2+1|\right) + \arctan(x) + \frac{1}{2(x^2+1)} - 3\underbrace{\int \frac{1}{\left(x^2+1\right)^2}\,dx}_{\text{Trig substitution}}$$

OUCH!

Note: $\int \frac{1}{\left(x^2+1\right)^2}\,dx = \int \frac{1}{\sec^4(\theta)} \sec^2(\theta)\,d\theta$

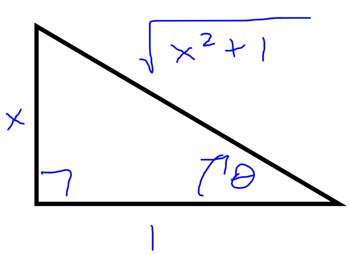
$= \int \cos^2(\theta)\,d\theta$

- $x = \tan(\theta)$
  $dx = \sec^2(\theta)\,d\theta$

$= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta) \right) d\theta$

- $\arctan(x) = \theta$

$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$



$= \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C$

$\sin(\theta) = \frac{x}{\sqrt{x^2+1}}$

$\cos(\theta) = \frac{1}{\sqrt{x^2+1}}$

$= \frac{1}{2}\arctan(x) + \frac{x}{2(x^2+1)} + C$

Compute $\int \frac{x^3 - 2x + 1}{x\left(x^2+1\right)^2} dx$

$$= \ln(|x|) - \frac{1}{2}\ln(|x^2+1|) + \arctan(x) + \frac{1}{2(x^2+1)} - 3\int \frac{1}{(x^2+1)^2} dx$$

$$= \ln(|x|) - \frac{1}{2}\ln(|x^2+1|) + \arctan(x) + \frac{1}{2(x^2+1)}$$

$$- 3\left(\frac{1}{2}\arctan(x) + \frac{x}{2(x^2+1)}\right) + C$$

$$= \ln(|x|) - \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\arctan(x) + \frac{1-3x}{x^2+1} + C$$