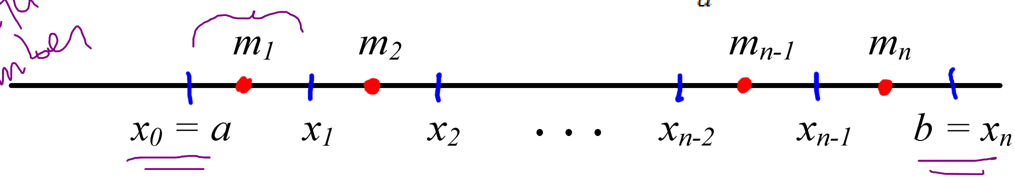
Final Comments on Numerical Integration

Methods for approximating $\int_a^b f(x)\,dx$ ✱

n is a natural number

$(b-a)/n$



crummy $$L_n = \frac{b-a}{n}\left[f(x_0)+f(x_1)+\cdots+f(x_{n-1})\right]$$

crummy $$R_n = \frac{b-a}{n}\left[f(x_1)+f(x_2)+\cdots+f(x_n)\right]$$

fair $$M_n = \frac{b-a}{n}\left[f\left(\underbrace{\frac{x_0+x_1}{2}}_{m_1}\right)+\cdots+f\left(\underbrace{\frac{x_{n-1}+x_n}{2}}_{m_n}\right)\right]$$

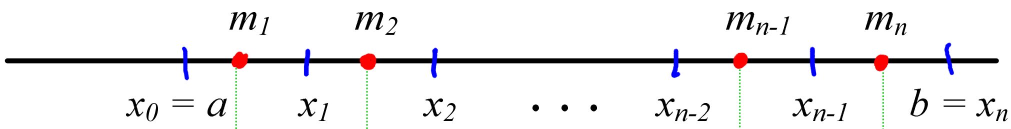
fair $$T_n = \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]$$

very good $$S_n = \frac{b-a}{6n}\left\{f(x_0)+f(x_n)+2\left[f(x_1)+\cdots+f(x_{n-1})\right]\right.$$
$$\left.+4\left[f\left(\underbrace{\frac{x_0+x_1}{2}}_{m_1}\right)+\cdots+f\left(\underbrace{\frac{x_{n-1}+x_n}{2}}_{m_n}\right)\right]\right\}$$

$$S_n = \frac{2}{3}M_n + \frac{1}{3}T_n$$

$$T_n = \frac{1}{2}L_n + \frac{1}{2}R_n$$

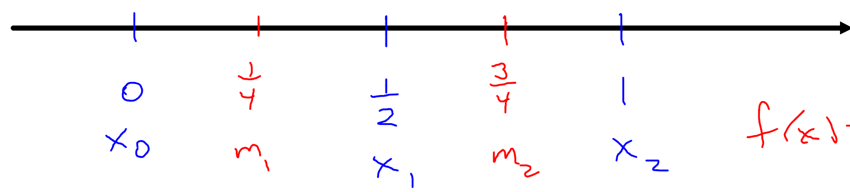
Methods for approximating $\int_a^b f(x)\,dx$



L_n	1		1		1	$\cdots$	1		1			$\frac{b-a}{n}$
R_n			1		1	$\cdots$	1		1		1	$\frac{b-a}{n}$
T_n	1		2		2	$\cdots$	2		2		1	$\frac{b-a}{2n}$
M_n		1		1		$\cdots$		1		1		$\frac{b-a}{n}$
S_n	1	4	2	4	2	$\cdots$	2	4	2	4	1	$\frac{b-a}{6n}$

See the videos posted on the course homepage.

Example: Use Simpson's method with $n = 2$ to approximate $\int_0^1 \sin\left(x^2\right) dx$.



$f(x) = \sin(x^2)$

$$S_n = \frac{1-0}{6 \cdot 2}\left[f(0) + 4f\left(\tfrac{1}{4}\right) + 2f\left(\tfrac{1}{2}\right) + 4f\left(\tfrac{3}{4}\right) + f(1)\right]$$

$$= \frac{1}{12}\left[0 + 4 \cdot \sin\left(\tfrac{1}{16}\right) + 2\sin\left(\tfrac{1}{4}\right) + 4\sin\left(\tfrac{9}{16}\right) + \sin(1)\right]$$

$$= \underline{\underline{0.3099}}439\ldots$$

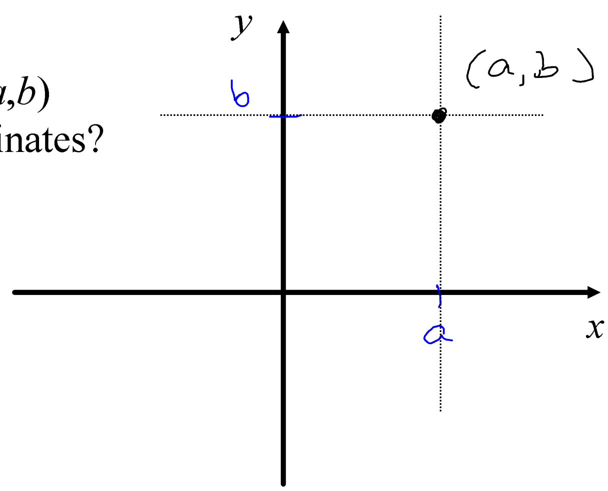
very close

Note: $\int_0^1 \sin(x^2)\,dx = \underline{\underline{0.310}}2683017$

calculator

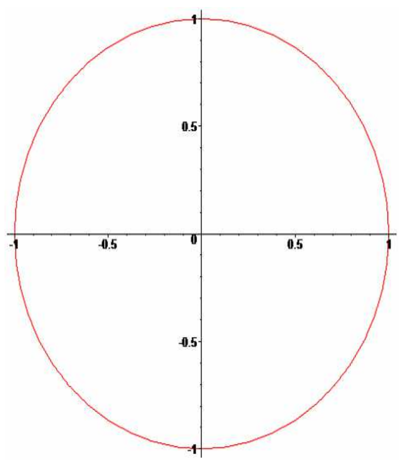
Polar Coordinates
(9.3 and 9.4)

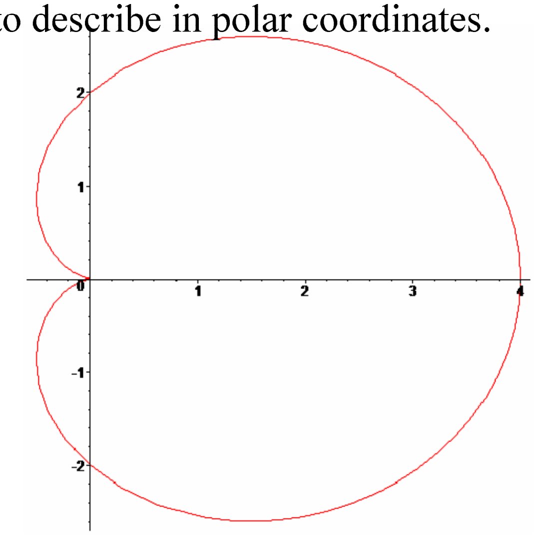
Question: How is the point (a,b) represented in cartesian coordinates?

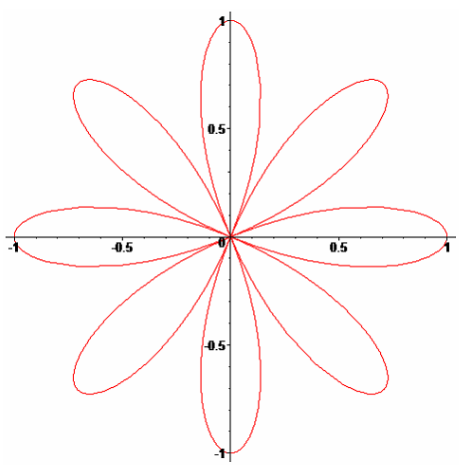


$\boxed{\text{Polar Coordinates}}$ afford another mechanism for visualizing points in the xy-plane.

The graphs below are easy to describe in polar coordinates.

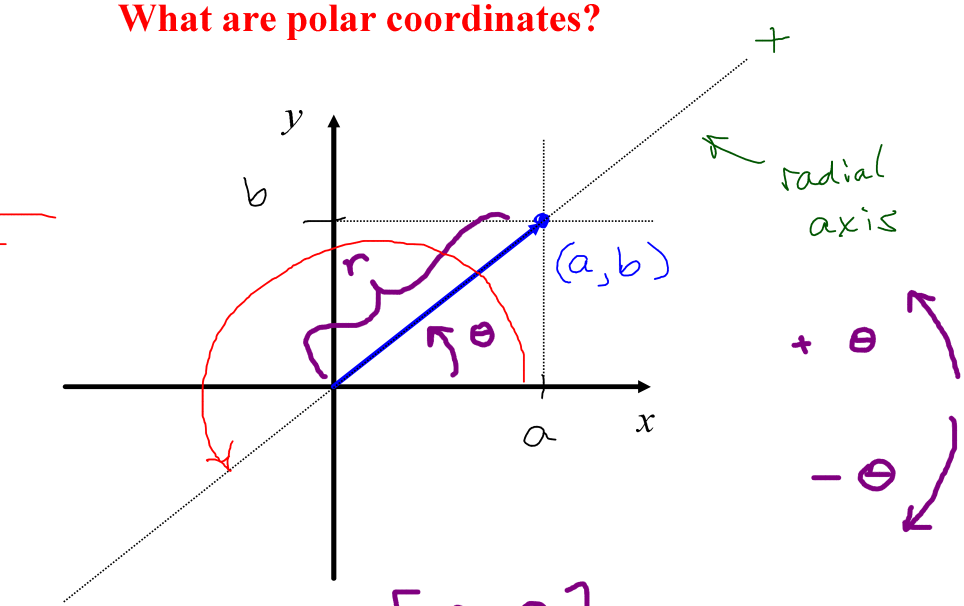






What are polar coordinates?

$$r = \sqrt{a^2 + b^2}$$

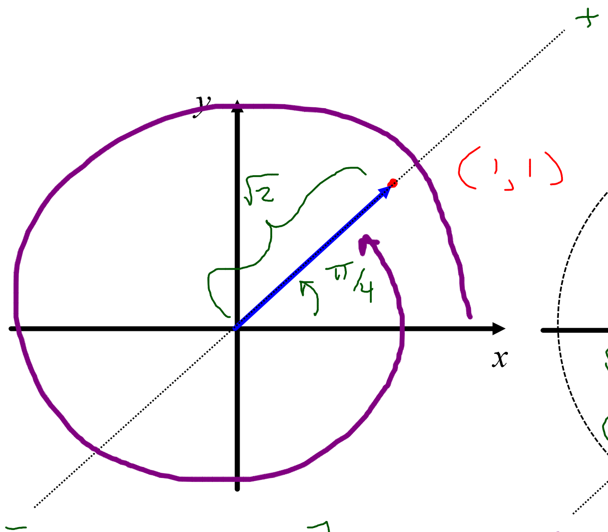


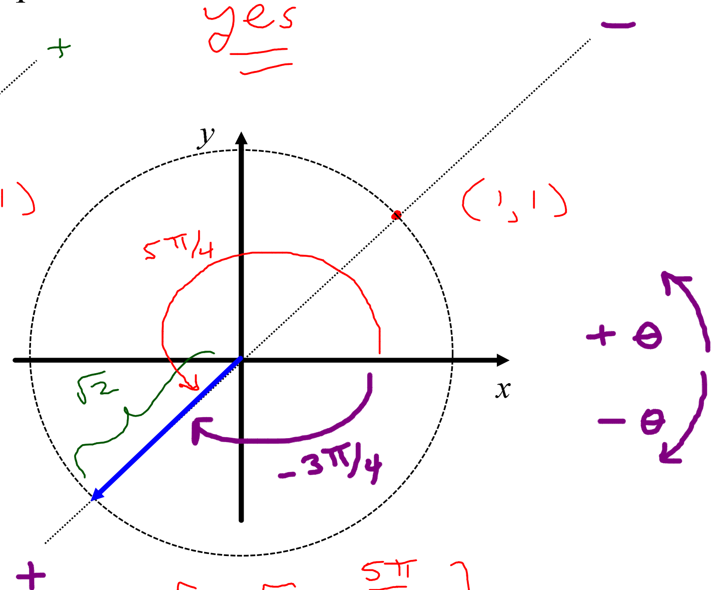
$[r, \theta]$

a polar representation of (a,b).

Question: Can we give more than one **polar representation** for the same point?

yes





$[\sqrt{2}, \frac{\pi}{4}]$ $\qquad$ $[-\sqrt{2}, \frac{5\pi}{4}]$

$[\sqrt{2}, \frac{\pi}{4} + 2\pi]$ $\qquad$ $[-\sqrt{2}, -\frac{3\pi}{4}]$

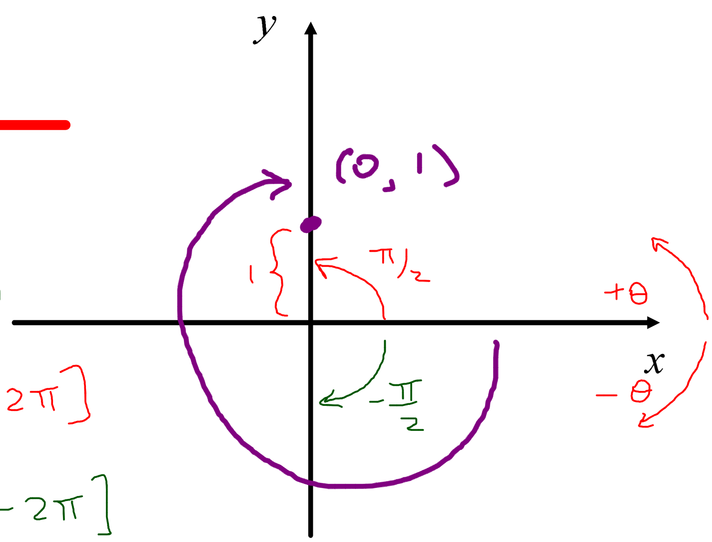
$[\sqrt{2}, \pi/4 + 4\pi]$ $\qquad$ $[-\sqrt{2}, -\frac{3\pi}{4} - 2\pi]$

$\vdots$ $\qquad$ $\vdots$

There are infinitely many polar representations for (1,1).

Example: How many ways can we represent the cartesian point (0,1) in polar coordinates? Infinitely many.

(x, y) rectangular	$[r, \theta]$ polar
(0,1)	$[1, \frac{\pi}{2}]$
	$[-1, -\frac{\pi}{2}]$
	$[1, \frac{\pi}{2} + 2\pi]$
	$[-1, -\frac{\pi}{2} - 2\pi]$
	$[1, -\frac{3\pi}{2}]$ and many more.



$[1, \frac{\pi}{2} + 2k\pi]$, k is an integer

$[-1, -\frac{\pi}{2} + 2k\pi]$, " " " "

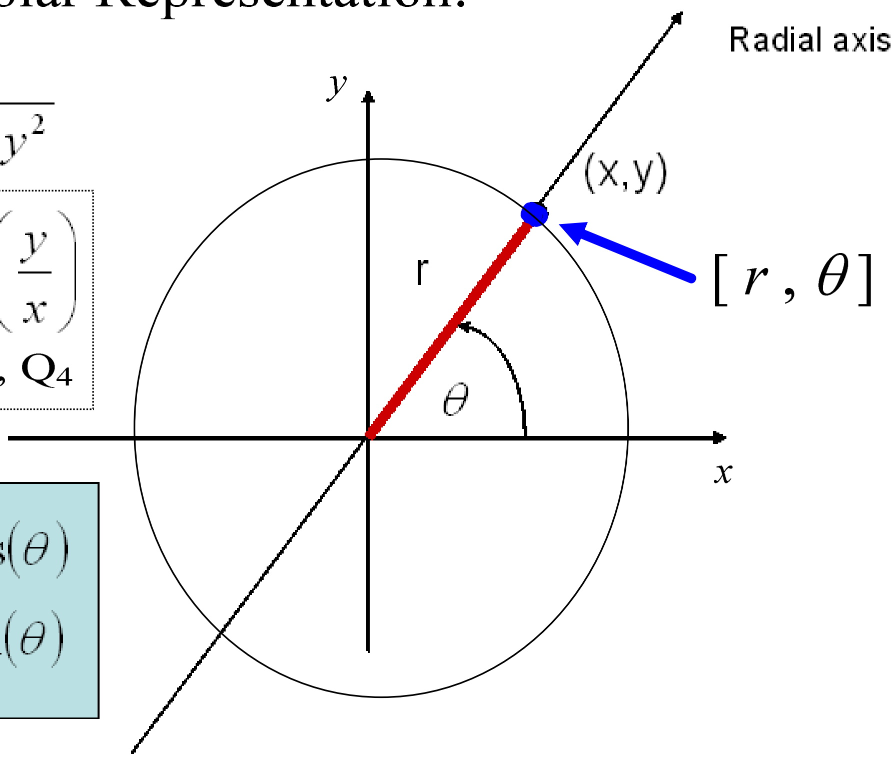
Standard Polar Representation:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

(x,y) in Q_1, Q_4

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$



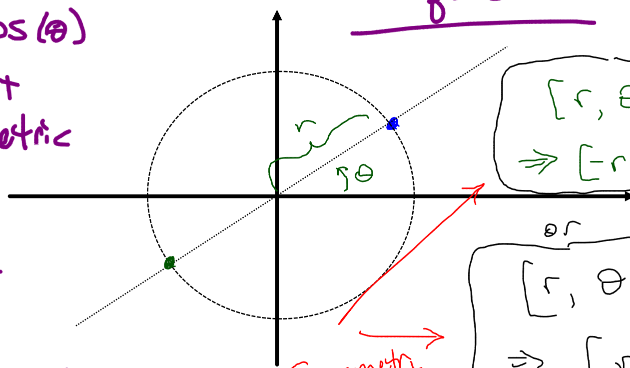
Example: Determine whether the polar graph of $\boxed{r = \cos(\theta)}$ is symmetric $\underbrace{\text{about the origin... the } x\text{-axis.}}_{\text{2 questions}}$

$r = \cos(\theta)$ is not symmetric about $(0, 0)$.

i.e.

$r = \cos(\theta)$

$\not\Rightarrow -r = \cos(\theta)$



$[r, \theta] \Rightarrow [-r, \theta]$

or

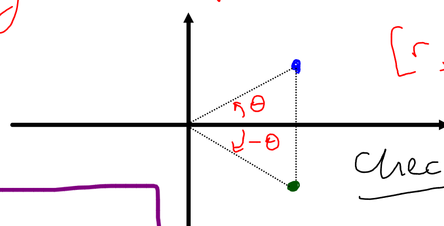
$[r, \theta] \Rightarrow [r, \theta + \pi]$

NO

symmetry about $(0,0)$

symmetry about the x-axis? Yes

$[r, \theta] \Rightarrow [r, -\theta]$.



Check: spse $[r, \theta]$ is on $r = \cos(\theta)$.

Note: $r = \cos(\theta) = \cos(-\theta)$ since cosine is an even function

$\Rightarrow [r, -\theta]$ is on the graph.

Note:

$r = \cos(\theta)$

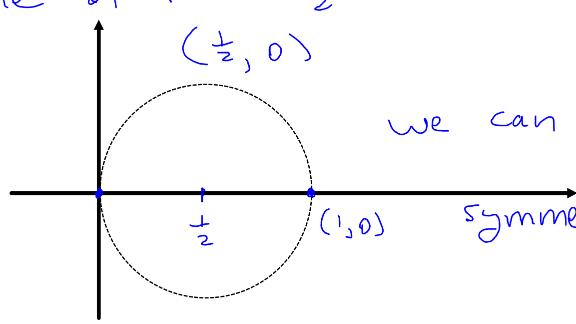
$r^2 = r\cos(\theta)$

$x^2 + y^2 = x$

$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4}$

$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4} = (\frac{1}{2})^2$

circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$



we can see the symmetry.

Rewrite the following curve in polar coordinates.

$$\boxed{\begin{array}{l} x = r\cos(\theta) \\ y = r\sin(\theta) \end{array}}$$

Easy

$$y = -\frac{1}{2}x^2 + 3x - 1$$

$$r\sin(\theta) = -\frac{1}{2}\left(r^2\cos^2(\theta)\right) + 3r\cos(\theta) - 1$$

Example: Rewrite $r = \sin(\theta)\tan(\theta)$ in cartesian coordinates.

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$x^2 + y^2 = r^2$

$$r \cdot r = r\sin(\theta)\frac{r\sin(\theta)}{r\cos(\theta)}$$

$$r^2 = r\sin(\theta) \cdot \frac{r\sin(\theta)}{r\cos(\theta)}$$

$$x^2 + y^2 = y \cdot \frac{y}{x}$$

$$\boxed{x^2 + y^2 = \frac{y^2}{x}}$$

Question: How can we describe lines in polar coordinates?

Cases: Horizontal, Vertical, $\boxed{\text{General}}$

$$y = a$$

$$r\sin(\theta) = a \Rightarrow \boxed{r = a\csc(\theta)}$$

$$x = a$$

$$r\cos(\theta) = a \Rightarrow \boxed{r = a\sec(\theta)}$$

$$ax + by = c$$

$$a\,r\cos(\theta) + b\,r\sin(\theta) = c$$

$$r = \frac{c}{a\cos(\theta) + b\sin(\theta)}$$