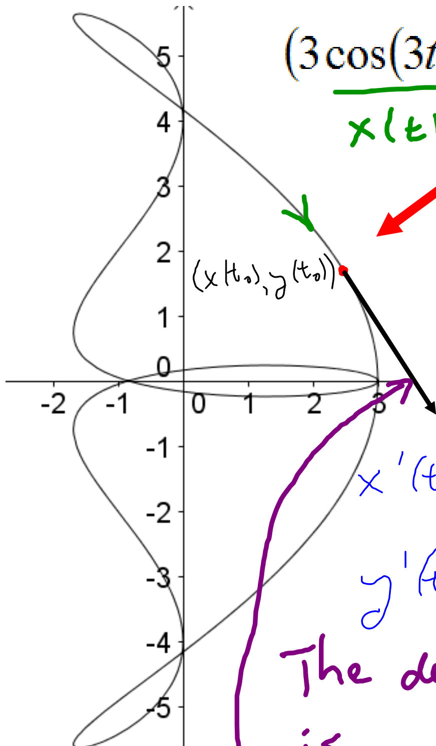
Parametric Curves
(continued)

Examples: Parametric curves can be interesting.

$\left(\cos(t), \sin\left(\sqrt{2}\, t\right)\right)$ ⟷ We saw this graph.

$\left(\underbrace{3\cos(3t)\cos(t)}_{x(t)}, \underbrace{(4-5\cos(t))\sin(t)}_{y(t)}\right)$



Question: What is the derivative in this setting?

$x'(t) = 3\cos(3t)(-\sin(t)) + \cos(t)(-9\sin(3t))$

$y'(t) = (4-5\cos(t))\cos(t) + \sin(t)\cdot 5\sin(t)$

The derivative of the parameterization is $x'(t)\,\underline{\underline{i}} + y'(t)\,\underline{\underline{j}}$.

Note: $\boxed{x'(t_0) i + y'(t_0) j}$ is tangent to the curve at $(x(t_0), y(t_0))$ and it points in the direction of orientation.

Note: If we have a parametric curve $(x(t), y(t))$ where $x(t)$ and $y(t)$ are differentiable functions, then we can compute $x'(t)$ and $y'(t)$.

Question: What do these derivatives represent?

1. See the previous page.

2. Slope: $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$

$$= \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

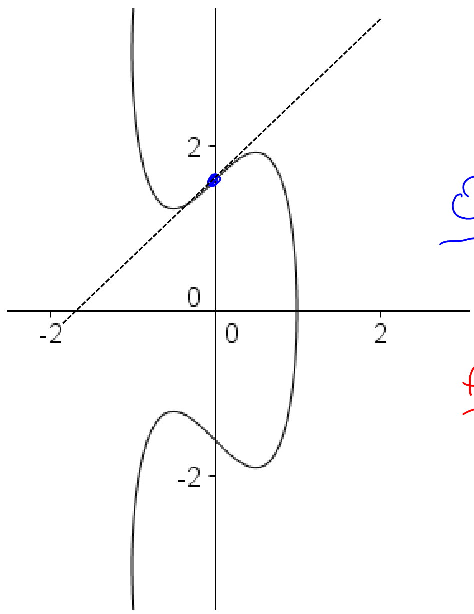
The slope of the curve at $(x(t_0), y(t_0))$ is $\dfrac{y'(t_0)}{x'(t_0)}$ provided $x'(t_0) \neq 0$.

Derivatives - The Complete Story...

1. If $x'(t_0) \neq 0$ then $\dfrac{y'(\mathrm{t}_0)}{x'(t_0)}$ is the slope of the tangent line to the curve at $(x(t_0), y(t_0))$.

2. If $x'(t_0)$ and $y'(t_0)$ are not both zero, then the vector $\boxed{x'(t_0)i + y'(t_0)j}$ is tangent to the curve at $(x(t_0), y(t_0))$, and it points in the direction of orientation along the curve.

Example: Give an equation for the tangent line to the curve parameterized by (cos(t) , t + sin(2t)) at the point (0,π/2).



$x(t) = \cos(t)$

$y(t) = t + \sin(2t)$

Q: What value of t places us at $(0, \pi/2)$?

A: Need $\cos(t) = 0$

$t + \sin(2t) = \frac{\pi}{2}$.

$\Rightarrow \boxed{t = \pi/2}$

$\Rightarrow$ slope of T.L. $= \dfrac{y'(\pi/2)}{x'(\pi/2)} = \dfrac{-1}{-1} = 1$

$x'(t) = -\sin(t)$, $y'(t) = 1 + 2\cos(2t)$

∴ The T.L. is

$$y - \frac{\pi}{2} = (x - 0)$$

i.e. $y = x + \pi/2$.

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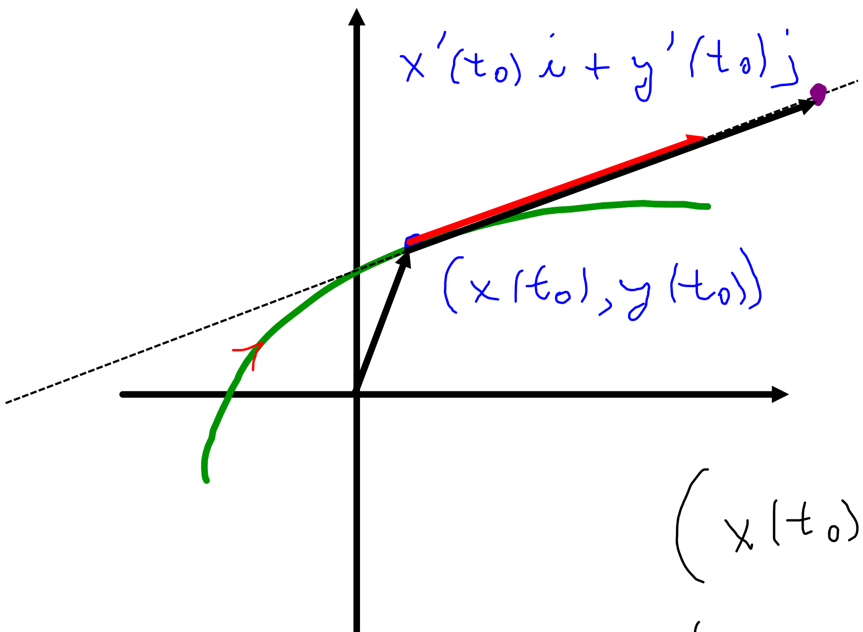
1. Give the slope of the tangent line to the curve given by the parametrization $\left(2t^2-1, \sin(t)\right)$, at the point where $t=1$.

2. Give the *y*-intercept of the tangent line to the curve given in the problem above at the point where $t=1$.

Question: How can we parameterize the tangent line at $(\,x(t_0)\,,\,y(t_0)\,)$ to the curve parameterized by $(\,x(t)\,,\,y(t)\,)$?



$$\underbrace{\left(x(t_0) + x'(t_0)\,t \;,\; y(t_0) + y'(t_0)\,t \right)}_{\text{a param. for the tangent line}}$$

Example: Give a parameterization of the tangent line to the graph of $(2\cos(t), 3\sin(t))$ at the point where $t = \pi/4$, and show the relationship between the vector $x'(\pi/4)\,\boldsymbol{i} + y'(\pi/4)\,\boldsymbol{j}$ and the graph of the parametric curve.

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$x(t) = 2\cos(t) \qquad x\left(\frac{\pi}{4}\right) = \sqrt{2}$

$y(t) = 3\sin(t) \qquad y\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

Ellipse!

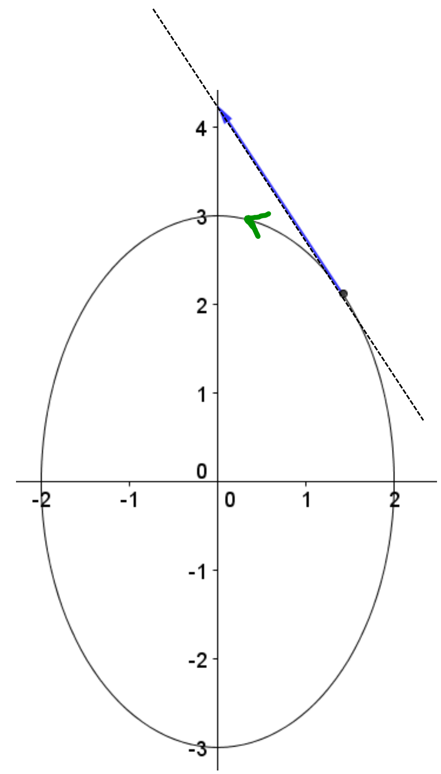
T.L.: Point $= \left(\sqrt{2}, \frac{3\sqrt{2}}{2}\right)$

$x'(t) = -2\sin(t) \Rightarrow x'\left(\frac{\pi}{4}\right) = -\sqrt{2}$

$y'(t) = 3\cos(t) \Rightarrow y'(\pi/4) = \frac{3\sqrt{2}}{2}$

Param for T.L.: $\left(\sqrt{2} + (-\sqrt{2})t, \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}t\right)$

Derivative Vector: $-\sqrt{2}\,i + \frac{3\sqrt{2}}{2}\,j$



Popper 12

3. Give the slope of the tangent line to the curve parameterized by $(\cos(t) + 2t\ ,\ \sin(2t))$ at the point where $t = 0$.

4. Give the slope of the normal line to the curve parameterized by $(t^3 - 2t\ ,\ 3t + 1)$ at the point $(4, -1)$.

3. $1/2$

4. 0