Parametric Curves
(continued)
**Examples:** Parametric curves can be interesting.

\[
\begin{pmatrix} \cos(t) \\ \sin\left(\sqrt{2} t\right) \end{pmatrix}
\]  
We saw this graph.

\[
\begin{pmatrix} 3\cos(3t)\cos(t) \\ (4 - 5\cos(t))\sin(t) \end{pmatrix}
\]  
\(x(t)\)  
\(y(t)\)

**Question:** What is the derivative in this setting?

**Note:**

\[
x'(t) = 3\cos(3t)(-\sin(t)) + \cos(t)(-9\sin(3t))
\]

\[
y'(t) = (4 - 5\cos(t))\cos(t) + \sin(t)(5\sin(t))
\]

The derivative of the parameterization is the vector

\[
x'(t)i + y'(t)j
\]
Note: If we have a parametric curve \((x(t), y(t))\) where \(x(t)\) and \(y(t)\) are differentiable functions, then we can compute \(x'(t)\) and \(y'(t)\).

**Question:** What do these derivatives represent?

1. \(x'(t_0)i + y'(t_0)j\) will be tangent to the curve at \((x(t_0), y(t_0))\).

2. **Slope:** \[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \cdot \frac{dt/dx}{dt/dt} = \frac{y'(t)}{x'(t)} \]

\[ \Rightarrow \frac{y'(t_0)}{x'(t_0)} = \text{slope of the tangent line to the curve at } (x(t_0), y(t_0)). \]
Derivatives - The Complete Story...

1. If \( x'(t_0) \neq 0 \) then \( \frac{y'(t_0)}{x'(t_0)} \) is the slope of the tangent line to the curve at \( (x(t_0), y(t_0)) \).

2. If \( x'(t_0) \) and \( y'(t_0) \) are not both zero, then the vector \( x'(t_0)i + y'(t_0)j \) is tangent to the curve at \( (x(t_0), y(t_0)) \), and it points in the direction of orientation along the curve.
Example: Give an equation for the tangent line to the curve parameterized by \((\cos(t), t + \sin(2t))\) at the point \((0, \pi/2)\).

We need \(\cos(t) = 0\)

and \(t + \sin(2t) = \frac{\pi}{2}\)

\(t = \frac{\pi}{2}\)

\[
\text{Slope} = \frac{y'(\frac{\pi}{2})}{x'(\frac{\pi}{2})} = \frac{-1}{-1} = 1.
\]

\(x'(t) = -\sin(t), y'(t) = 1 + 2\cos(2t)\)
**Question:** How can we parameterize the tangent line at \((x(t_0), y(t_0))\) to the curve parameterized by \((x(t), y(t))\)?

\[
\begin{align*}
\frac{y - y(t_0)}{x - x(t_0)} &= \frac{y'(t_0) t}{x'(t_0) t} \\
\text{slope} &= \frac{y'(t_0)}{x'(t_0)}
\end{align*}
\]
Example: Give a parameterization of the tangent line to the graph of \((2\cos(t),3\sin(t))\) at the point where \(t = \frac{\pi}{4}\), and show the relationship between the vector \(x'(\frac{\pi}{4})\mathbf{i} + y'(\frac{\pi}{4})\mathbf{j}\) and the graph of the parametric curve.

\[
\begin{align*}
x(t) &= 2\cos(t) \\
y(t) &= 3\sin(t)
\end{align*}
\]

\[
\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1
\]

i.e., \[
\frac{x^2}{4} + \frac{y^2}{9} = 1
\]

**ellipse!**

\[
x'(t) = -2\sin(t) \Rightarrow x'(\frac{\pi}{4}) = -\sqrt{2}
\]

\[
y'(t) = 3\cos(t) \Rightarrow y'(\frac{\pi}{4}) = \frac{3\sqrt{2}}{2}
\]

**Note:** slope of T.L. is \(-\frac{3}{2}\).

and a parameterization for the T.L. is given by

\[
\begin{align*}
x &= \sqrt{2} + (-\sqrt{2})t, \\
y &= \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}t
\end{align*}
\]

**Derivative:**

\[-\sqrt{2} \mathbf{i} + \frac{3\sqrt{2}}{2} \mathbf{j}
\]

\[
(\sqrt{2} - \sqrt{2}t, \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}t)
\]