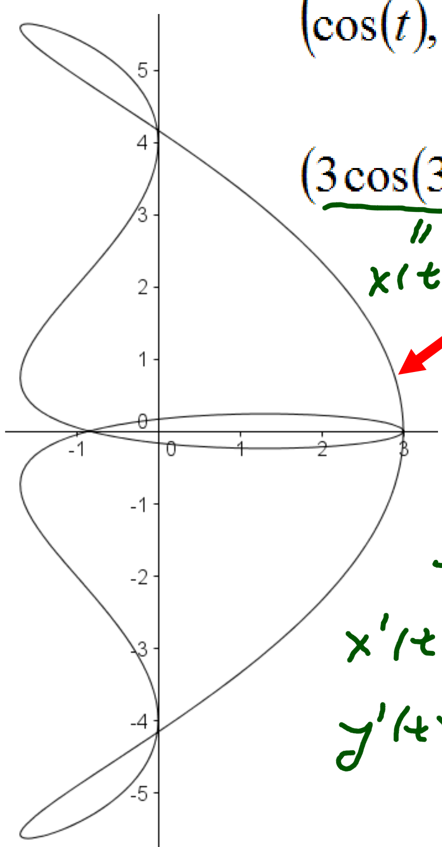
Parametric Curves
(continued)

Examples: Parametric curves can be interesting.



$$\left(\cos(t), \sin\left(\sqrt{2}\,t\right)\right) \longleftrightarrow$$ We saw this graph.

$$\left(\underbrace{3\cos(3t)\cos(t)}_{=\,x(t)}, \underbrace{(4-5\cos(t))\sin(t)}_{=\,y(t)}\right)$$

Question: What is the derivative in this setting?

Note:

$$x'(t) = 3\cos(3t)(-\sin(t)) + \cos(t)(-9\sin(3t))$$

$$y'(t) = (4-5\cos(t))\cos(t) + \sin(t)\,5\sin(t)$$

The derivative of the parameterization is the vector

$$x'(t)\,i + y'(t)\,j.$$

Note: If we have a parametric curve $(\ x(t), y(t)\ )$ where $x(t)$ and $y(t)$ are differentiable functions, then we can compute $x\,'(t)$ and $y\,'(t)$.

Question: What do these derivatives represent?

1. $x'(t_0)\,i + y'(t_0)\,j$ will be tangent to the curve at $(x(t_0), y(t_0))$.

2. Slope: $\dfrac{dy}{dx} = \dfrac{dy/dt \cdot dt/dx}{dx/dt \cdot dt/dx} = \dfrac{y'(t)}{x'(t)}$

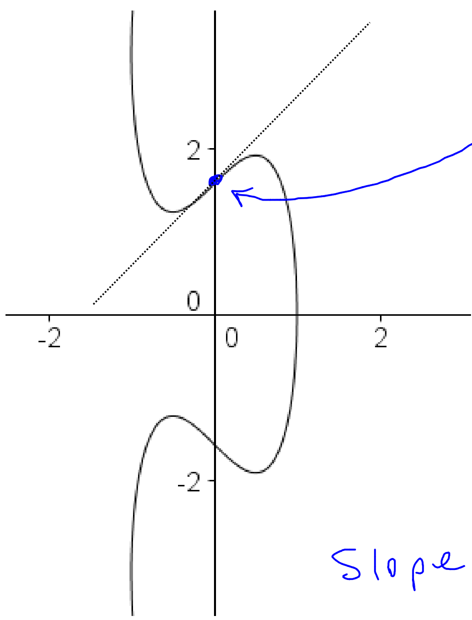
$\Rightarrow \dfrac{y'(t_0)}{x'(t_0)}$ = slope of the tangent line to the curve at $(x(t_0), y(t_0))$.

Derivatives - The Complete Story...

1. If $x'(t_0) \neq 0$ then $\dfrac{y'(t_0)}{x'(t_0)}$ is the slope of the tangent line to the curve at $(x(t_0), y(t_0))$.

2. If $x'(t_0)$ and $y'(t_0)$ are not both zero, then the vector $x'(t_0)i + y'(t_0)j$ is tangent to the curve at $(x(t_0), y(t_0))$, and it points in the direction of orientation along the curve.

Example: Give an equation for the tangent line to the curve parameterized by ($\cos(t)$, $t + \sin(2t)$) at the point $(0,\pi/2)$.



$t = ?$

we need

$\cos(t) = 0$

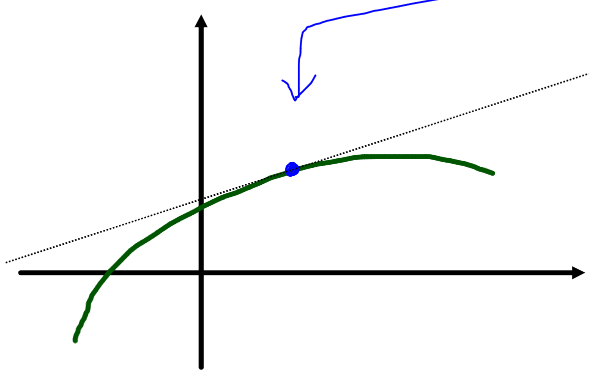
and $t + \sin(2t) = \frac{\pi}{2}$

$t = \pi/2$

Slope $= \dfrac{y'(\pi/2)}{x'(\pi/2)} = \dfrac{-1}{-1} = 1.$

$x'(t) = -\sin(t)$, $y'(t) = 1 + 2\cos(2t)$

Question: How can we parameterize the tangent line at $(\,x(t_0)\,,\,y(t_0)\,)$ to the curve parameterized by $(\,x(t)\,,\,y(t)\,)$?



$$\boxed{\begin{aligned} x &= x(t_0) + x'(t_0)\,t \\ y &= y(t_0) + y'(t_0)\,t \end{aligned}}$$

Note: $$\frac{y - y(t_0)}{x - x(t_0)} = \frac{y'(t_0)\,\cancel{t}}{x'(t_0)\,\cancel{t}} = \text{slope}$$

Example: Give a parameterization of the tangent line to the graph of $\boxed{(2\cos(t), 3\sin(t))}$ at the point where $t = \pi/4$, and show the relationship between the vector $x'(\pi/4)\,\boldsymbol{i} + y'(\pi/4)\,\boldsymbol{j}$ and the graph of the parametric curve.

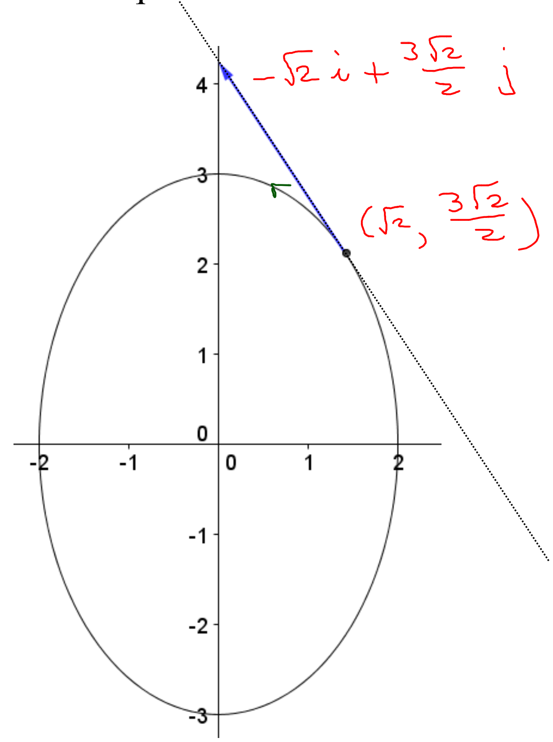
$x(t) = 2\cos(t)$
$y(t) = 3\sin(t)$

$x(\frac{\pi}{4}) = \sqrt{2}$
$y(\frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$

$(\frac{x}{2})^2 + (\frac{y}{3})^2 = 1$

i.e. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

ellipse!



$x'(t) = -2\sin(t) \Rightarrow x'(\frac{\pi}{4}) = -\sqrt{2}$
$y'(t) = 3\cos(t) \Rightarrow y'(\frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$

Note: slope of T.L. is $-\frac{3}{2}$.

and a parameterization for the T.L. is given by $\boxed{x = \sqrt{2} + (-\sqrt{2})t,\ y = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}t}$

$\Updownarrow$

$(\sqrt{2} - \sqrt{2}t, \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}t)$

Derivative:
$-\sqrt{2}\,i + \frac{3\sqrt{2}}{2}\,j$