Popper 14

1. Give the slope of the tangent line to the polar curve $r = 1 + \cos(\theta)$ at the point where $\theta = \pi/2$.

2. Give the x-intercept of the tangent line in problem 1.

New Material

Position, velocity, speed and acceleration of a particle.

position at time t $\rightarrow$ $r(t) = (x(t), y(t))$

position vector at time t. $\rightarrow$ $\overrightarrow{r}(t) = x(t)i + y(t)j$

velocity at time t $\rightarrow$ $\overrightarrow{v}(t) = x'(t)i + y'(t)j \leftarrow r'(t)$

speed $\rightarrow$ $\left|\overrightarrow{v}(t)\right| = \sqrt{(x'(t))^2 + (y'(t))^2}$

Acceleration?

$\vec{a}(t) = \dfrac{d\vec{v}(t)}{dt} = r''(t)$

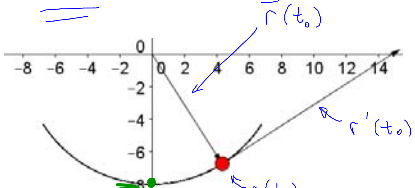
$= x''(t)i + y''(t)j$

Example: A position of a particle at time t is given by ($8\sin(\cos(2t))$, $-8\cos(\cos(2t))$). Discuss the motion of the particle. What is the position when the speed of the particle as large as possible?

$\vec{v}(t) = r'(t) = 8\cos(\cos(2t))\cdot(-2\sin(2t))i + 8\sin(\cos(2t))(-2\sin(2t))j$

$r(t)$

$\vec{r}(t_0)$



$r'(t_0)$

$r(t_0)$

Oscillates : See the video.

Speed $= |\vec{v}(t)| = \sqrt{(-16\cos(\cos(2t))\sin(2t))^2 + (-16\sin(\cos(2t))\sin(2t))^2}$

$= \sqrt{(-16\sin(2t))^2} = 16|\sin(2t)|$

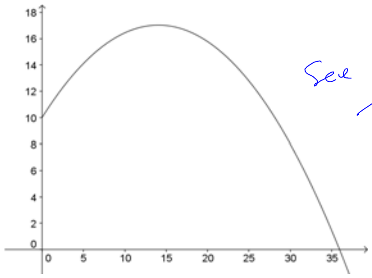
Max value of speed is 16 when $t = \pi/4$, $3\pi/4$, etc...

Position when $t = \frac{\pi}{4}$: $(0, -8)$.

Falling Bodies
(neglecting friction)

See the video

Example: An object is launched from a height of 10 ft, at an angle of $\pi/4$ radians to horizontal. If the initial speed is 30 ft/sec, when will the object strike the ground, and what will the velocity of the object be at impact?



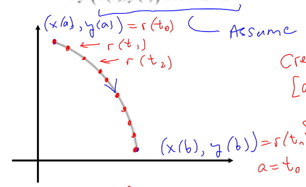
See the video

Length of a Curve

Popper 14
3. 72
4. 0
5. 0
6. 25

$r(t)$

Goal: Find the length of the curve parameterized by $\underbrace{(x(t), y(t))}$ for $a \le t \le b$.



← Assume 1-1.

Create a partition for $[a,b]$ in the form

$a = t_0 < t_1 < t_2 < \cdots < t_n = b$

$$\text{length} \approx \sum_{i=1}^{n} \left| r(t_{i-1})\, r(t_i) \right|$$

sum of the lengths of small line segments

$$= \sum_{i=1}^{n} \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$$

$$= \sum_{i=1}^{n} \sqrt{\Bigg(\underbrace{\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}}_{\approx x'(t_i)}\Bigg)^2 + \Bigg(\underbrace{\frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}}}_{\approx y'(t_i)}\Bigg)^2}\ \underbrace{(t_i - t_{i-1})}_{\Delta t_i}$$

$$\approx \sum_{i=1}^{n} \sqrt{x'(t_i)^2 + y'(t_i)^2}\ \Delta t_i$$

R.S. for $\displaystyle\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$

⇑ Exact length of the curve

If a curve C is parameterized by $(x(t), y(t))$ for $a \le t \le b$, and both $x'(t)$ and $y'(t)$ are continuous functions, then the length of the curve is given by

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

(assuming no backtracking or duplication occurs)

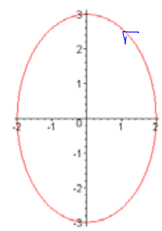
Note: If the parameterization above gives the position of a particle at time t, then the integral formula above gives the total distance travelled by the particle for $a \le t \le b$. In this case, you do not need to worry about backtracking or duplication. The formula gives the total distance travelled.

Example: Give an integral representing the length of the curve given parametrically by

$$(\underbrace{2\cos(t)}_{x(t)}, \underbrace{3\sin(t)}_{y(t)})$$

for $0 \le t \le 2\pi$.

← one revolution



$x'(t) = -2\sin(t)$

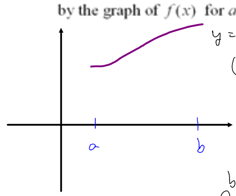
$y'(t) = 3\cos(t)$

$$\text{length} = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2(t) + 9\cos^2(t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{4 + 5\cos^2(t)}\, dt$$

Question: How can we find the length of the curve given by the graph of $f(x)$ for $a \le x \le b$?



$y = f(x)$

$\hookrightarrow$ param $\quad (\underbrace{t}_{x(t)}, \underbrace{f(t)}_{y(t)})$

$a \le t \le b.$

$$\text{Length} = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_a^b \sqrt{1 + f'(t)^2}\, dt$$

$$= \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

Try $f(x) = x^3$ for $0 \le x \le 1$.

$$\int_0^1 \sqrt{1 + (3x^2)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 9x^4}\, dx$$

Circle of radius r $\quad (r\cos(t), r\sin(t))$

$0 \le t \le 2\pi$

$$\text{Length} = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2 \sin^2(t) + r^2 \cos^2(t)}\, dt$$

$$= \int_0^{2\pi} r\, dt = rt \Big|_0^{2\pi}$$

$$= 2\pi r.$$