New Material

Position, velocity, speed and acceleration of a particle.

Position at time t. $\longrightarrow$ $r(t)=(x(t),y(t))$

Position vector $\longrightarrow$ $\vec{r}(t)=x(t)i+y(t)j$

velocity $\longrightarrow$ $\vec{v}(t)=x'(t)i+y'(t)j = r'(t)$

Speed $\longrightarrow$ $\left|\vec{v}(t)\right|=\sqrt{(x'(t))^2+(y'(t))^2}$

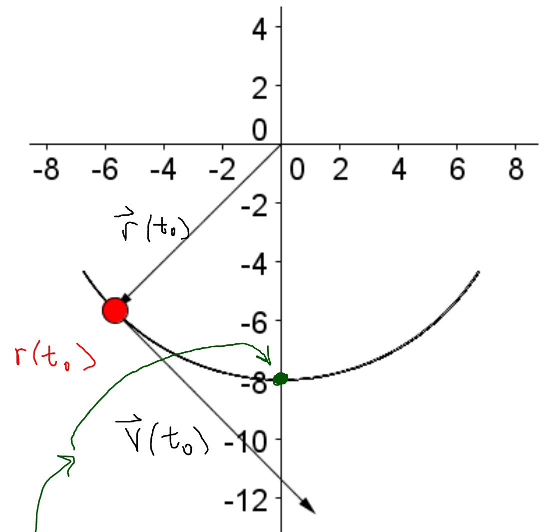
Acceleration?

$$r''(t) = x''(t)\,i + y''(t)\,j$$

Example: A position of a particle at time t is given by

$*$ ($8\sin(\cos(2t))$, $-8\cos(\cos(2t))$). Discuss the motion of the particle. What is the position when the speed of the particle as large as possible?

$r(t)$ $\quad \vec{V}(t) = r'(t) = 8\cos(\cos(2t))(-2\sin(2t))\, i + 8\sin(\cos(2t))(-2\sin(2t))\, j$



$$\text{Speed} = |\vec{V}(t)| = \sqrt{(-16\cos(\cos(2t))\sin(2t))^2 + (-16\sin(\cos(2t))\sin(2t))^2}$$

$$= \sqrt{(16\sin(2t))^2 \left[\underbrace{\cos^2(\cos(2t)) + \sin^2(\cos(2t))}_{1} \right]}$$

$$= 16\,|\sin(2t)|$$

Note: The max value of speed is 16. It occurs when $\sin(2t) = \pm 1$.

e.g. $\pi/4$, $\frac{3\pi}{4}$, ...

Recall: $r(t) =$ ($8\sin(\cos(2t))$, $-8\cos(\cos(2t))$)

Note: $\sin(2t) = \pm 1 \Rightarrow \cos(2t) = 0$

$$\Rightarrow r(t) = (8\sin(0), -8\cos(0))$$

$$= (0, -8).$$

"Falling Bodies"

*(neglecting friction)

$g\,j$

$$g \approx -9.8 \ m/sec^2$$

$$g \approx -32 \ ft/sec^2$$

Accelleration is constant.

$$r''(t) = \vec{a}(t) = 0\,i + g\,j$$

$$\vec{v}(t) = c_1\,i + (gt + c_2)\,j$$

Note: when $t=0$, $\vec{v}(0) = \underbrace{c_1\,i + c_2\,j}_{\text{Known, if initial velocity is known.}}$

$$\vec{r}(t) = (c_1 t + c_3)\,i + \left(\tfrac{1}{2} g t^2 + c_2 t + c_4\right) j$$

Note: when $t=0$

$$\vec{r}(0) = \underbrace{c_3\,i + c_4\,j}_{\text{known if initial position is known.}}$$

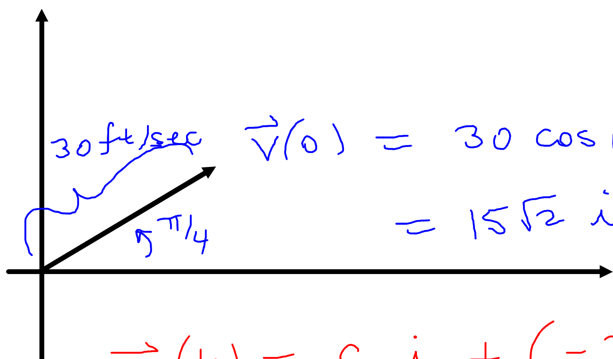
Example: An object is launched from a height of 10 ft, at an angle of $\pi/4$ radians to horizontal. If the initial speed is 30 ft/sec, when will the object strike the ground, and what will the velocity of the object be at impact? (neglect friction)

$$\vec{r}(t) = (c_1 t + c_3)\,i + \left(\tfrac{1}{2} g t^2 + c_2 t + c_4\right) j$$

$$\vec{r}(0) = 0\,i + 10\,j$$

$$\Rightarrow c_3 = 0, \quad c_4 = 10$$

$$\vec{r}(t) = c_1 t\, i + \left(-16t^2 + c_2 t + 10\right) j$$



$$\vec{v}(0) = 30\cos\left(\tfrac{\pi}{4}\right) i + 30\sin(\pi/4)\, j$$

$$= 15\sqrt{2}\, i + 15\sqrt{2}\, j$$

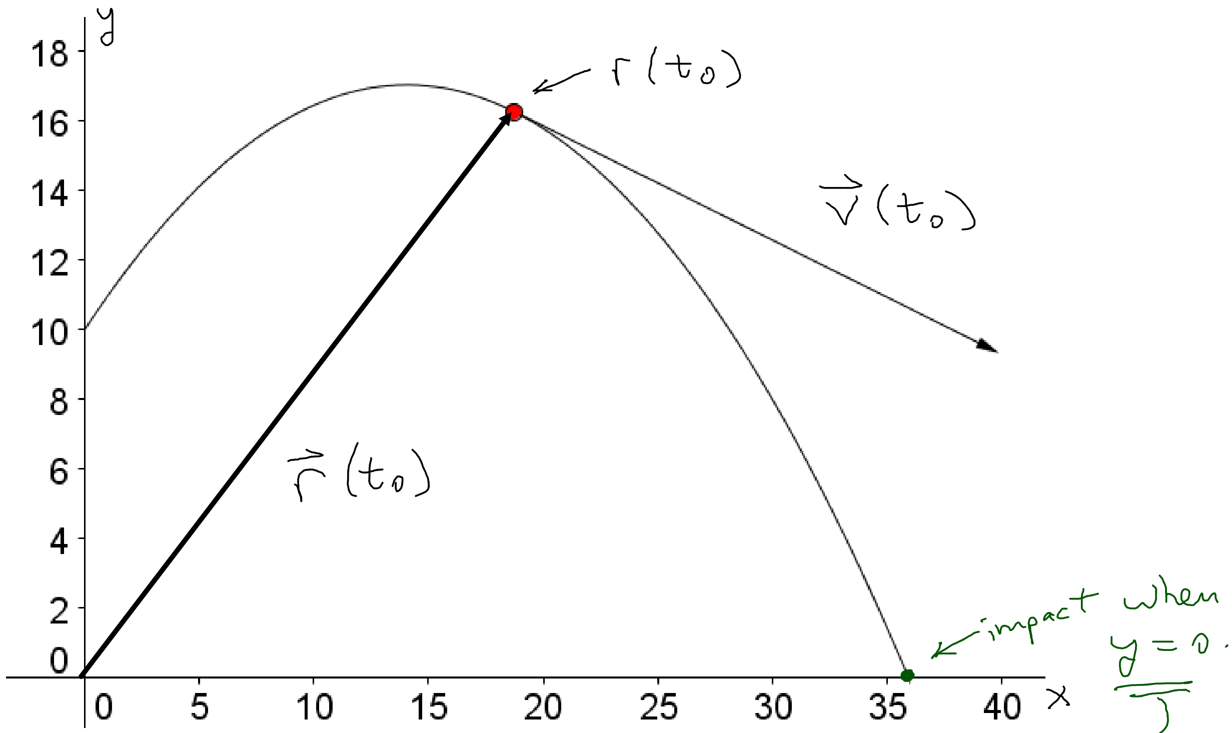
$$\vec{v}(t) = c_1\, i + (-32t + c_2)\, j$$

$$\Rightarrow c_1 = 15\sqrt{2}, \quad c_2 = 15\sqrt{2}$$

$$\Rightarrow$$

$$\vec{r}(t) = 15\sqrt{2}\, t\, i + \left(-16t^2 + 15\sqrt{2}\, t + 10\right) j$$

Note: This is only valid up to the time of impact.



$$-16t^2 + 15\sqrt{2}\,t + 10 = 0, \quad t \geq 0$$

Use quad. formula $\quad t \approx$ **1.6949359836** seconds

To get the velocity at impact, evaluate

$$\vec{v}(t) = 15\sqrt{2}\,i + (-32t + 15\sqrt{2})\,j$$

at $t =$ **1.6949359836**

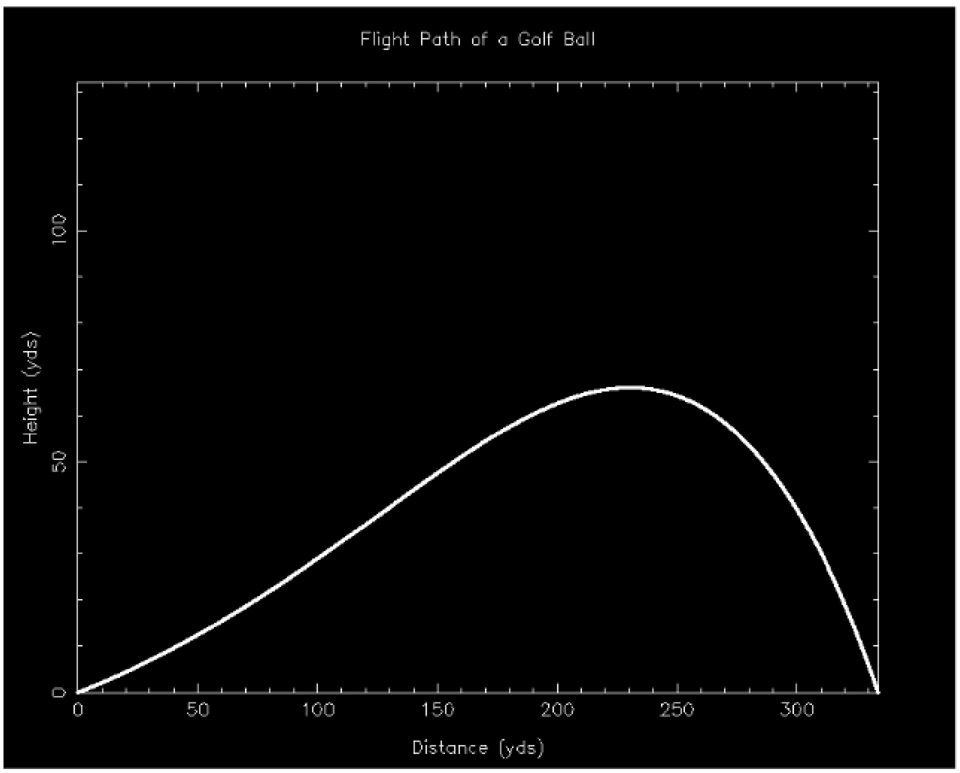
$$\Rightarrow \vec{v}(\mathbf{1.6949359836}) = 15\sqrt{2}\,i + (\mathbf{-33.0247480396})\,j$$

ft/sec.

$$\text{Speed} = \sqrt{(15\sqrt{2})^2 + (\mathbf{-33.0247480396})^2}$$

$$= \mathbf{39.2509106019} \text{ ft/sec.}$$

Note: Lift and Drag can play a significant role in the flight path of objects at high velocity. For example, consider the path shown below for a golf ball struck by a club with speed 125mph at an angle of 8.5° with the horizontal.

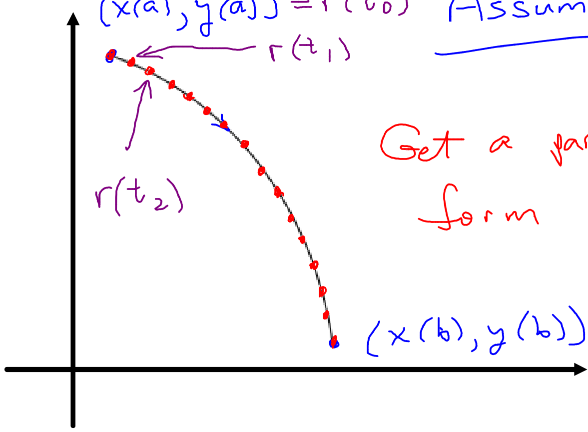


An interesting talk explaining the mathematics of the flight of golf balls and other objects can be found at http://www.youtube.com/watch?v=e6v9ib-dOtg. You can also find this video by doing a search for "Doug Arnold, golf".

Length of a Curve

$r(t)$

Goal: Find the length of the curve parameterized by $\boxed{\left(x(t), y(t)\right)}$ for $a \le t \le b$.



Assume: The param. is one to one.

Get a partion of $[a,b]$ in the form $a = t_0 < t_1 < t_2 < \cdots < t_n = b$

$$\text{length} \approx \sum_{i=1}^{n} \left| \overline{r(t_{i-1})\, r(t_i)} \right|$$

$$= \sum_{i=1}^{n} \sqrt{\left(x(t_i) - x(t_{i-1})\right)^2 + \left(y(t_i) - y(t_{i-1})\right)^2}$$

$$= \sum_{i=1}^{n} \sqrt{\left(\underbrace{\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}}_{\approx x'(t_i)}\right)^2 + \left(\underbrace{\frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}}}_{\approx y'(t_i)}\right)^2}\; \underbrace{(t_i - t_{i-1})}_{\Delta t_i}$$

$$\approx \sum_{i=1}^{n} \sqrt{x'(t_i)^2 + y'(t_i)^2}\; \Delta t_i$$

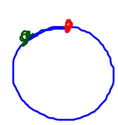
R.S. for $\displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

If a curve C is parameterized by $(x(t), y(t))$ for $a \le t \le b$, and both $x'(t)$ and $y'(t)$ are continuous functions, then the length of the curve is given by

$$\int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

Assuming no backtracking or duplication occurs.
i.e. the parameterization is one-to-one.

* **Note:** If the parameterization above gives the position of a particle at time t, then the integral formula above gives the total distance travelled by the particle for $a \le t \le b$. In this case, you do not need to worry about backtracking or duplication. The formula gives the total distance travelled.



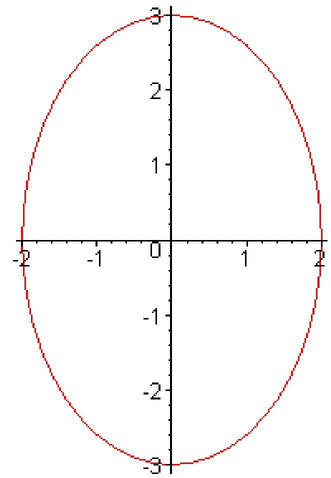
Warning: This integral is typically a beast!! Except for very special cases, numerical integration has to be used to compute the value.

Example: Give an integral representing the length of the curve given parametrically by

$$\Big(\underbrace{2\cos(t)}_{x(t)},\underbrace{3\sin(t)}_{y(t)}\Big)$$

for $0 \le t \le 2\pi$.

one revolution



$x'(t) = -2\sin(t)$

$y'(t) = 3\cos(t)$

$$\text{Length} = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2(t) + 9\cos^2(t)}\, dt = \int_0^{2\pi} \sqrt{4 + 5\cos^2(t)}\, dt$$

You must use numeric integration to get this value.

Example: Verify the formula for the circumference of a circle.

Param for circle of radius $r > 0$.

$$\big(\underbrace{r\cos(t)}_{x(t)},\ \underbrace{r\sin(t)}_{y(t)}\big) \qquad 0 \le t \le 2\pi.$$

$$x'(t) = -r\sin(t)$$

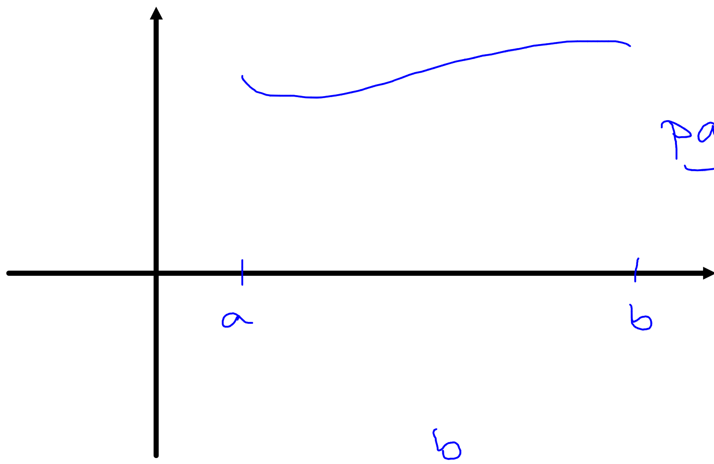
$$y'(t) = r\cos(t)$$

$$\text{Circumference} = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2\sin^2(t) + r^2\cos^2(t)}\, dt$$

$$= \int_0^{2\pi} r\, dt = rt\Big|_0^{2\pi} = 2\pi r$$

Question: How can we find the length of the curve given by the graph of $f(x)$ for $a \le x \le b$?



param $(t, f(t))$ $a \le t \le b$

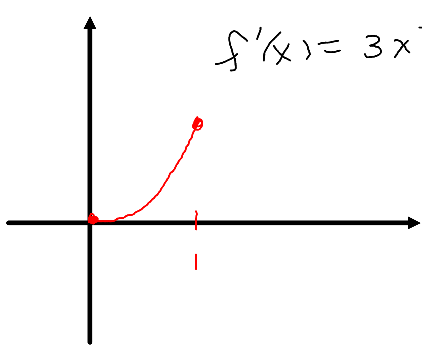
$$\text{Length} = \int_a^b \sqrt{1^2 + (f'(t))^2}\, dt$$

$$= \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Note: You typically have to compute this numerically.

ex. $f(x) = x^3, \quad 0 \le x \le 1.$

$f'(x) = 3x^2$



$$\text{Length} = \int_0^1 \sqrt{1 + (3x^2)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 9x^4}\, dx$$

Numeric integration is needed here.