Information

Practice Test 3 is posted.

You should have already registered for Test 3.

Pie a Prof - Tuesday at 2pm

Questions:

1. What is a sequence?

① A list of numbers (starting point can vary)

② A function from $\{1,2,3,\ldots\}$ to $\mathbb{R}$.

2. What is the limit of a sequence?

A value, if one exists, that the values tend towards.

Notation: $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ or $\{b_n\}_{n=0}^{\infty}$ or a_n.

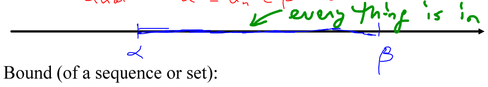
ex. $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$

New: Terminology for both sequences and sets.

1. Bounded Sequence (or set): A sequence a_n is bounded iff there are numbers α and β so that $\alpha \le a_n \le \beta$ for all n.

everything is in here



2. Upper Bound (of a sequence or set):

A number that is $\geq$ everything in the sequence (set).

3. Least Upper Bound (of a sequence or set): LUB

smallest upper bound.

4. Lower Bound (of a sequence or set):

A number that is $\le$ everything in the sequence (set).

5. Greatest Lower Bound (of a sequence or set): GLB

largest lower bound

Example: Give the LUB and GLB of the interval [-2,3).

LUB: 3

GLB: -2

$\{x \mid -2 \le x < 3\}$

Example: Give the LUB and GLB of the interval $\{x \mid x^2 - x < 2\}$. $\equiv S$

"The set of all x such that $x^2 - x < 2$"

$x^2 - x < 2$

$x^2 - x - 2 < 0$

parab. turning up.

Determine $x^2 - x - 2 = 0$.

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$

$\therefore S = (-1, 2)$.

LUB is 2

GLB is -1

Popper 16

1. Give the LUB of $\{x \mid x^2 - 1 < 3\}$.

Example: Give the LUB and GLB of $\left\{1 - \frac{2}{n}\right\}_{n=1}^{\infty}$.

Look at some terms.

$-1, 0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \ldots$

"increasing" with limit 1

GLB is -1

LUB is 1

Popper 16

2. Give the GLB of $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$.

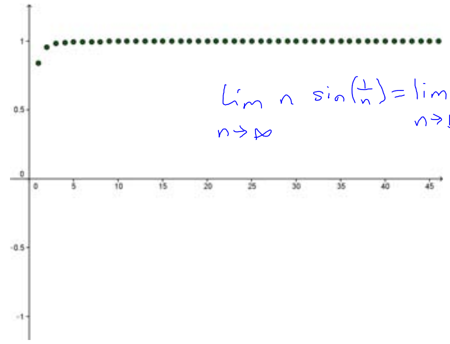
3. Give the limit (if it exists) of $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$.

Example: Give the limit (if it exists) of $\left\{\dfrac{\sin(n)}{n}\right\}_{n=1}^{\infty}$.

0

Example: Give the limit of $\left\{n\sin\left(\dfrac{1}{n}\right)\right\}$.

$\lim\limits_{u\to 0}\dfrac{\sin(u)}{u}=1$



$\lim\limits_{n\to\infty} n\sin\left(\frac{1}{n}\right)=\lim\limits_{n\to\infty}\dfrac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}=1$

Popper 16

4. Give the limit (if it exists) of $\left\{\dfrac{1+\sin(n)}{n}\right\}_{n=1}^{\infty}$.

Terminology: $\{a_n\}$

strict

1. Increasing Sequence:

$a_k < a_{k+1}$ for all k.

2. Non-decreasing Sequence:

$a_k \le a_{k+1}$ for all k

strict

3. Decreasing Sequence

$a_k > a_{k+1}$ for all k

4. Non-increasing Sequence:

$a_k \ge a_{k+1}$ for all k.

Note: A sequence is **monotone** if and only if it is either increasing, nondecreasing, decreasing or nonincreasing.

Question: What tool can be used to help determine whether a sequence is increasing or decreasing?

Derivative!

ex: $\left\{ 1 - \frac{2}{n} \right\}_{n=1}^{\infty}$

$f(x) = 1 - \frac{2}{x}$, $x \geq 1$

$f'(x) = \frac{2}{x^2} > 0$

$\Rightarrow f$ is increasing

$\Rightarrow \left\{ 1 - \frac{2}{n} \right\}_{n=1}^{\infty}$ is increasing.

Facts:

If a sequence has a limit, then it is bounded. Warning: The converse is not necessarily true!!

If a sequence is increasing, then the GLB is the first term and the LUB is the limit (if it exists).

If a sequence is decreasing, then the GLB is limit (if it exists) and the LUB is the first term.

Example: Determine whether the sequence $\left\{ 1 + \frac{3n+1}{n+2} \right\}_{n=1}^{\infty}$ is

a. Bounded Yes since the sequence has a limit.

b. Monotone Yes b/c it is increasing.

Then, give the limit (if it exists).

Note: If a sequence has a limit, then it is bounded.

$\underbrace{1 + \boxed{\frac{3n+1}{n+2}}}_{\to 4}$, $\frac{3n+1}{n+2} \to 3$

Limit? yes $\boxed{4}$

Note: $\frac{3n+1}{n+2} = \frac{n\left(3 + \frac{1}{n}\right)}{n\left(1 + \frac{2}{n}\right)} = \frac{3 + \frac{1}{n}}{1 + \frac{2}{n}} \to 3.$ ($\frac{1}{n} \to 0$, $\frac{2}{n} \to 0$)

underlying function:

$f(x) = 1 + \frac{3x+1}{x+2}$, $x \geq 1$

$f'(x) = \frac{(x+2)\cdot 3 - (3x+1)\cdot 1}{(x+2)^2}$

$= \frac{5}{(x+2)^2} > 0$

$\therefore f$ is increasing for $x \geq 1$

$\Rightarrow$ sequence is increasing.

Example: Determine whether the sequence $\left\{\frac{3n+(-1)^n}{n+2}\right\}_{n=1}^{\infty}$ is

a. Bounded ✓

b. Monotone NO

Then, give the limit (if it exists).

$$\frac{n}{n}\cdot\frac{3+\frac{(-1)^n}{n}}{1+\frac{2}{n}}$$

Note: $\lim\limits_{n\to\infty}\frac{3n+(-1)^n}{n+2}=3$

$\therefore$ since there is a limit, the sequence is bounded.

Example: Determine whether the sequence $\left\{\frac{\sqrt{n+1}}{\sqrt{n}}\right\}_{n=1}^{\infty}$ is

a. Bounded ✓ yes, since the limit exists.

b. Monotone ✓ yes decreasing.

Then, give the limit (if it exists).

Note: $\lim\limits_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n}}=\lim\limits_{n\to\infty}\sqrt{\frac{n+1}{n}}=1$

?

$f(x)=\sqrt{\frac{x+1}{x}}, \quad x\geq 1$

$$f'(x)=\frac{1}{2\sqrt{\frac{x+1}{x}}}\cdot\frac{x-(x+1)}{x^2}$$

$$=\underbrace{\frac{1}{2\sqrt{\frac{x+1}{x}}}}_{\geq 0}\cdot\underbrace{\left(\frac{-1}{x^2}\right)}_{<0}$$

$\therefore f'(x)<0$ for $x\geq 1$

$\Rightarrow f$ is decreasing $\Rightarrow$ sequence is decreasing.

Example: Determine whether the sequence $\{1-e^{-2n}\}_{n=1}^{\infty}$ is

a. Bounded yes, since the limit exists.

b. Monotone yes. increasing!

Then, give the limit (if it exists).

$\lim\limits_{n\to\infty}(1-e^{-2n})=1$

$f(x)=1-e^{-2x}, \quad x\geq 1$

$f'(x)=2e^{-2x}>0$

$\therefore f$ is increasing $\Rightarrow$ sequence is increasing.

Example: Give the limit (if it exists) of $\left\{\left(\frac{2}{n}\right)^n\right\}_{n=1}^{\infty}$.

Terms $\left\{2, 1, \frac{8}{27}, \frac{1}{16}, \ldots\right\}$

Guess = Limit is 0.

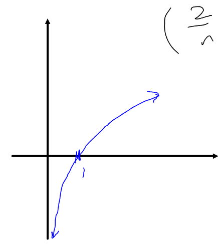
Why? 2 answers.

1. Note $0<\frac{2}{n}\leq\frac{2}{3}$ for $n\geq 3$.

$0<\left(\frac{2}{n}\right)^n\leq\left(\frac{2}{3}\right)^n$ for $n\geq 3$, $\left(\frac{2}{3}\right)^n\to 0$

$\therefore$ limit is 0.

2. Use logs.



$$\left(\frac{2}{n}\right)^n=e^{\ln\left(\left(\frac{2}{n}\right)^n\right)}$$

$$=e^{\underbrace{n\ln\left(\frac{2}{n}\right)}_{\to-\infty}}$$

$$=e^{\text{"}-\infty\text{"}}$$

$$=0$$

Example: Give the limit (if it exists) of $\left\{\frac{2n^2-3n+6}{3n-16n^2+12}\right\}_{n=1}^{\infty}$.

$$\lim_{n\to\infty} \frac{2n^2-3n+6}{3n-16n^2+12} = -\frac{1}{8}$$

Note:

$$\frac{2n^2-3n+6}{3n-16n^2+12} = \frac{n^2}{n^2}\cdot\frac{\left(2-\frac{3}{n}+\frac{6}{n^2}\right)}{\left(\frac{3}{n}-16+\frac{12}{n^2}\right)}$$

$$= \frac{2-\underbrace{\frac{3}{n}}_{\to 0}+\underbrace{\frac{6}{n^2}}_{\to 0}}{\underbrace{\frac{3}{n}}_{\to 0}-16+\underbrace{\frac{12}{n^2}}_{\to 0}} \to \frac{2}{-16} = -\frac{1}{8}$$

Example: Give the limit (if it exists) of $\left\{n^n\right\}_{n=1}^{\infty}$.

$$\lim_{n\to\infty} n^n = \infty$$

diverges to ∞.

Example: Give the limit (if it exists) of $\left\{\ln(n+1)-\ln(n)\right\}_{n=1}^{\infty}$.

$$\boxed{\lim_{n\to\infty}\left(\ln(n+1)-\ln(n)\right)=0}$$

$\ln(n+1) \to \infty$, $\ln(n) \to \infty$

$\infty - \infty$??

wth?

Note:

$\ln(n+1)-\ln(n)$

$$= \ln\left(\underbrace{\frac{n+1}{n}}_{\to 1}\right) \to \ln(1) = 0.$$