Information

Practice Test 3 is posted.

You should have already registered for Test 3.

Pie a Prof - Tuesday at 2pm

Questions:

1. What is a sequence?

① A list of numbers ← starting point can vary

② A function from $\{1, 2, 3, \ldots\}$ to $\mathbb{R}$.

2. What is the limit of a sequence?

A value, if one exists, that the values tend towards.

Notation: $\{a_n\}$

or $\{a_n\}_{n=1}^{\infty}$

or $\{b_n\}_{n=0}^{\infty}$

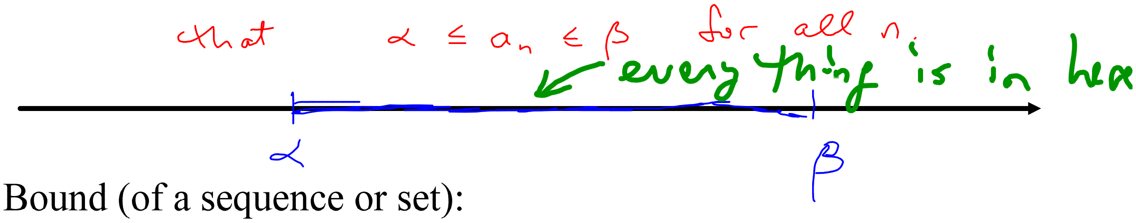
or a_n.

ex: $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$

New: Terminology for both sequences and sets.

1. Bounded Sequence (or set): A sequence a_n is bounded iff there are numbers α and β so that $\alpha \le a_n \le \beta$ for all n.



2. Upper Bound (of a sequence or set):

A number that is $\ge$ everything in the sequence (set).

3. Least Upper Bound (of a sequence or set): LUB

smallest upper bound.

4. Lower Bound (of a sequence or set):

A number that is $\le$ everything in the sequence (set).

5. Greatest Lower Bound (of a sequence or set): GLB

largest lower bound

Example: Give the LUB and GLB of the interval [-2,3).

3

-2

$\{x \mid -2 \leq x < 3\}$

Example: Give the LUB and GLB of the interval $\{ x \mid x^2 - x < 2 \}. \equiv S$

"The set of all x such that $x^2 - x < 2$"

$x^2 - x < 2$

$x^2 - x - 2 < 0$

parab. turning up.

Determine $x^2 - x - 2 = 0$.

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$

$\therefore S = (-1, 2)$.

LUB is 2

GLB is -1

Popper 16

1. Give the LUB of $\left\{x \mid x^2 - 1 < 3\right\}$.

Example: Give the LUB and GLB of $\left\{1-\frac{2}{n}\right\}_{n=1}^{\infty}$.

Look at some terms.

$-1, 0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \ldots$

"increasing" with limit 1

GLB is -1

LUB is 1

Popper 16

2. Give the GLB of $\left\{ \dfrac{(-1)^n}{n} \right\}_{n=1}^{\infty}$.

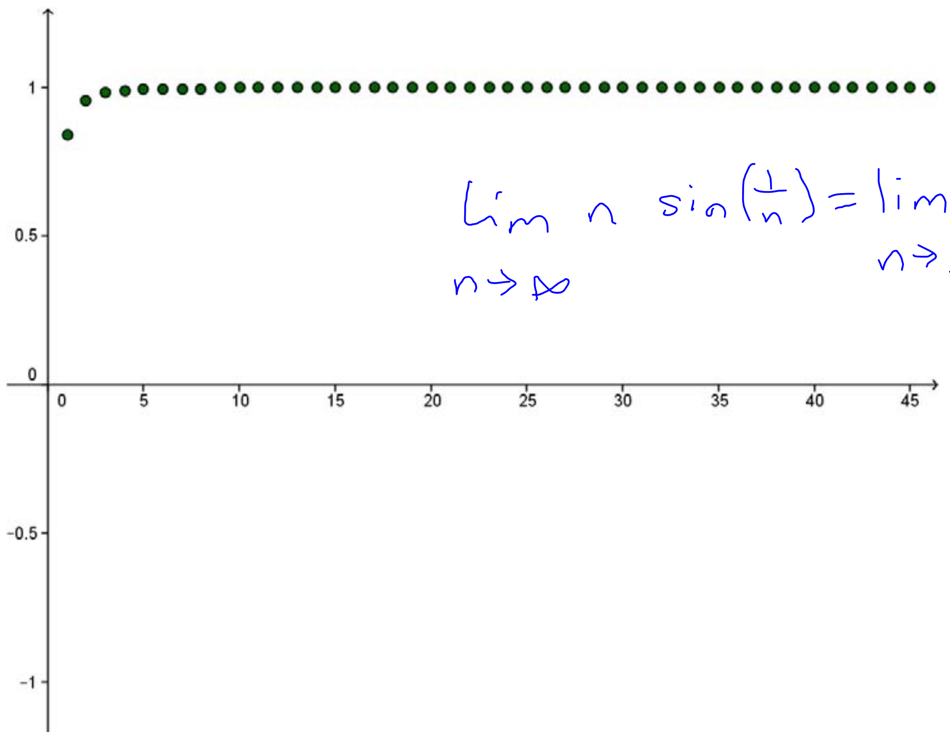
3. Give the limit (if it exists) of $\left\{ \dfrac{(-1)^n}{n} \right\}_{n=1}^{\infty}$.

Example: Give the limit (if it exists) of $\left\{ \frac{\sin(n)}{n} \right\}_{n=1}^{\infty}$.

0

Example: Give the limit of $\left\{ n \sin\left(\frac{1}{n}\right) \right\}$.

$$\lim_{u \to 0} \frac{\sin(u)}{u} = 1$$



$$\lim_{n \to \infty} n \sin\left(\frac{1}{n}\right) = \lim_{n \to \infty} \frac{\sin\left(\overset{\to 0}{\frac{1}{n}}\right)}{\underset{\to 0}{\frac{1}{n}}} = 1$$

Popper 16

4. Give the limit (if it exists) of $\left\{\dfrac{1+\sin(n)}{n}\right\}_{n=1}^{\infty}$.

Terminology: $\{ a_n \}$

strict

1. Increasing Sequence:

$$a_k < a_{k+1} \quad \text{for all } k.$$

2. Non-decreasing Sequence:

$$a_k \leq a_{k+1} \quad \text{for all } k$$

strict

3. Decreasing Sequence

$$a_k > a_{k+1} \quad \text{for all } k$$

4. Non-increasing Sequence:

$$a_k \geq a_{k+1} \quad \text{for all } k.$$

Note: A sequence is **monotone** if and only if it is either increasing, nondecreasing, decreasing or nonincreasing.

Question: What tool can be used to help determine whether a sequence is increasing or decreasing?

Derivative!

ex:

$$\left\{1-\frac{2}{n}\right\}_{n=1}^{\infty}$$

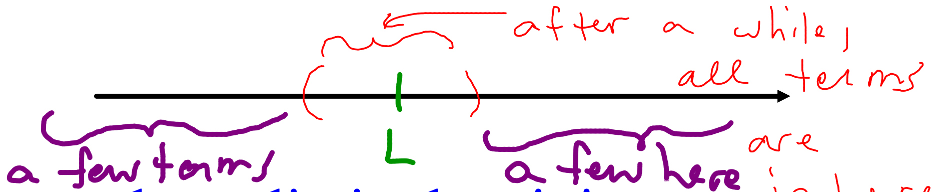
$$f(x)=1-\frac{2}{x}, \quad x\geq 1$$

$$f'(x)=\frac{2}{x^2}>0$$

$\Rightarrow$ f is increasing

$\Rightarrow$ $\left\{1-\frac{2}{n}\right\}_{n=1}^{\infty}$ is increasing.

Facts:



If a sequence has a limit, then it is bounded. Warning: The converse is not necessarily true!!

If a sequence is increasing, then the GLB is the first term and the LUB is the limit (if it exists).

If a sequence is decreasing, then the GLB is limit (if it exists) and the LUB is the first term.

Example: Determine whether the sequence $\left\{1+\frac{3n+1}{n+2}\right\}_{n=1}^{\infty}$ is

a. Bounded Yes since the sequence has a limit.

b. Monotone Yes b/c it is increasing.

Then, give the limit (if it exists).

Note: If a sequence has a limit, then it is bounded.

$$\underbrace{1+\boxed{\frac{3n+1}{n+2}}}_{\to 4} \qquad \frac{3n+1}{n+2} \to 3$$

Limit? yes $\boxed{4}$

Note: $\frac{3n+1}{n+2} = \frac{n\left(3+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)} = \frac{3+\frac{1}{n}}{1+\frac{2}{n}} \to 3.$ ($\frac{1}{n} \to 0$, $\frac{2}{n} \to 0$)

underlying function:

$$f(x) = 1 + \frac{3x+1}{x+2}, \quad x \geq 1$$

$$f'(x) = \frac{(x+2)\cdot 3 - (3x+1)\cdot 1}{(x+2)^2} = \frac{5}{(x+2)^2} > 0$$

$\therefore$ f is increasing for $x \geq 1$

$\Rightarrow$ sequence is increasing.

Example: Determine whether the sequence $\left\{\frac{3n+(-1)^n}{n+2}\right\}_{n=1}^{\infty}$ is

a. Bounded ✓

b. Monotone NO

Then, give the limit (if it exists).

$$\frac{n}{n}\cdot\frac{3+\frac{(-1)^n}{n}}{1+\frac{2}{n}}$$

Note:

$$\lim_{n\to\infty}\frac{3n+(-1)^n}{n+2}=3$$

$\therefore$ since there is a limit, the sequence is bounded.

Example: Determine whether the sequence $\left\{ \frac{\sqrt{n+1}}{\sqrt{n}} \right\}_{n=1}^{\infty}$ is

a. Bounded ✓ yes, since the limit exists.

b. Monotone ✓ yes decreasing.

Then, give the limit (if it exists).

Note! $\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}}$ $\quad \frac{n+1}{n} \to 1$

$= 1$

?

$$f(x) = \sqrt{\frac{x+1}{x}}, \quad x \geq 1$$

$$f'(x) = \frac{1}{2\sqrt{\frac{x+1}{x}}} \cdot \frac{x-(x+1)}{x^2}$$

$$= \underbrace{\frac{1}{2\sqrt{\frac{x+1}{x}}}}_{\geq 0} \cdot \underbrace{\left(\frac{-1}{x^2}\right)}_{< 0}$$

$\therefore f'(x) < 0$ for $x \geq 1$

$\Rightarrow f$ is decreasing $\Rightarrow$ sequence is decreasing.

Example: Determine whether the sequence $\left\{1-e^{-2n}\right\}_{n=1}^{\infty}$ is

a. Bounded yes, since the limit exists.

b. Monotone yes. increasing!

Then, give the limit (if it exists).

$$\lim_{n\to\infty}\left(1-e^{-2n}\right)=1$$

$$f(x)=1-e^{-2x}, \quad x\geq 1$$

$$f'(x)=2e^{-2x}>0$$

$\therefore$ f is increasing

$\Rightarrow$ sequence is increasing.

Example: Give the limit (if it exists) of $\left\{\left(\frac{2}{n}\right)^n\right\}_{n=1}^{\infty}$.

Terms $\left\{2, 1, \frac{8}{27}, \frac{1}{16}, \ldots\right\}$

Guess = Limit is 0.

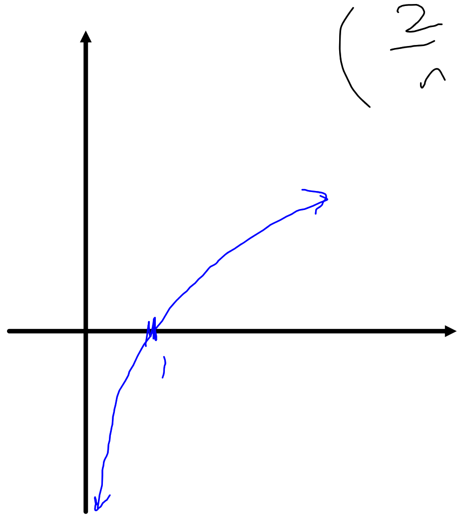
Why? 2 answers.

1. Note $0 < \frac{2}{n} \le \frac{2}{3}$ for $n \geq 3$.

$$0 < \left(\frac{2}{n}\right)^n \le \underbrace{\left(\frac{2}{3}\right)^n}_{\to 0} \quad \text{for } n \geq 3$$

$\therefore$ limit is 0.

2. Use logs.



$$\left(\frac{2}{n}\right)^n = e^{\ln\left(\left(\frac{2}{n}\right)^n\right)}$$

$$= e^{\underbrace{n \underbrace{\ln \big(\underbrace{\tfrac{2}{n}}_{\to 0^+}\big)}_{\to -\infty}}_{\to -\infty}}$$

$$= e^{\text{“}-\infty\text{”}}$$

$$= 0$$

Example: Give the limit (if it exists) of $\left\{ \dfrac{2n^2 - 3n + 6}{3n - 16n^2 + 12} \right\}_{n=1}^{\infty}$.

$$\lim_{n \to \infty} \frac{2n^2 - 3n + 6}{3n - 16n^2 + 12} = -\frac{1}{8}$$

Note:

$$\frac{2n^2 - 3n + 6}{3n - 16n^2 + 12} = \frac{n^2}{n^2} \cdot \frac{\left(2 - \frac{3}{n} + \frac{6}{n^2}\right)}{\left(\frac{3}{n} - 16 + \frac{12}{n^2}\right)}$$

$$= \frac{2 - \overset{\to 0}{\frac{3}{n}} + \overset{\to 0}{\frac{6}{n^2}}}{\underset{\to 0}{\frac{3}{n}} - 16 + \underset{\to 0}{\frac{12}{n^2}}} \to \frac{2}{-16} = -\frac{1}{8}$$

Example: Give the limit (if it exists) of $\left\{n^n\right\}_{n=1}^{\infty}$.

$$\lim_{n\to\infty} n^n = \infty$$

diverges to ∞.

Example: Give the limit (if it exists) of $\{\ln(n+1)-\ln(n)\}_{n=1}^{\infty}$.

$$\boxed{\lim_{n\to\infty}\left(\ln(n+1)-\ln(n)\right)=0}$$

$\ln(n+1) \to \infty$, $\ln(n) \to \infty$

$\infty - \infty$??

wth?

Note:

$$\ln(n+1)-\ln(n)$$

$$=\ln\left(\frac{n+1}{n}\right) \to \ln(1)=0.$$

$\frac{n+1}{n} \to 1$