Equivalent Statements:

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1. 3.7

The sequence converges. = The sequence has a limit.

The sequence diverges. = The sequence has no limit.

Language: $a_n \to L$ or $\{a_n\} \to L$

"a_n converges to L"

"a_n has limit L"

or $\{a_n\}_{n=1}^{\infty} \to L$

$0! = 1,\ 1! = 1,\ 2! = 2,\ 3! = 6,\ 4! = 24,$

for $n \geq 2$, $n! = n \cdot (n-1)! = n \cdot (n-1) \cdot \dots \cdot 1$.

Important Limits:

For each $\alpha > 0$ (fixed) $\frac{1}{n^\alpha} \to 0$ as $n \to \infty$.	For each real x (fixed) $\frac{x^n}{n!} \to 0$ as $n \to \infty$.
If $\lvert x \rvert < 1$, then $x^n \to 0$ as $n \to \infty$.	If $\varepsilon > 0$ then $\frac{\ln n}{n^\varepsilon} \to 0$ as $n \to \infty$. (see 2 pages forward)
If $x > 0$ (fixed), then $x^{1/n} \to 1$ as $n \to \infty$.	$n^{1/n} \to 1$ as $n \to \infty$. $e^{\ln(n^{1/n})} = e^{\frac{\ln(n)}{n}} \to e^0 = 1$

For each real x

$$\left(1 + \frac{x}{n}\right)^n \to e^x \quad \text{as} \quad n \to \infty.$$

From chapter 7.

Comment on growth rates

$$\ln(n) << n^{\varepsilon} << x^n << n! << n^n$$

$\varepsilon > 0$, $x > 1$

Note: $a_n << b_n$ means $\frac{a_n}{b_n} \to 0$.

$\varepsilon > 0$

$\frac{\ln n}{n^\varepsilon} \to 0$ as $\boxed{n \to \infty.}$ Why?

(from the definition of $\ln(n)$)

First: $\frac{\ln(n)}{n} = \frac{1}{n} \int_1^n \frac{1}{x}\,dx$

Spse $n > 1$.

$\leq \frac{1}{n} \int_1^n \frac{1}{\sqrt{x}}\,dx$

b/c $\frac{1}{\sqrt{x}} > \frac{1}{x}$ for $x > 1$.

$= \frac{1}{n} 2\sqrt{x}\Big|_1^n = \frac{2}{n}(\sqrt{n} - 1)$

$= \frac{2\sqrt{n} - 2}{n} \to 0$

So, for $n \geq 1$

$0 \leq \frac{\ln(n)}{n} \leq \frac{2\sqrt{n} - 2}{n} \to 0$

pinched!

$\therefore \lim_{n \to \infty} \frac{\ln(n)}{n} = 0.$

Note: $\varepsilon > 0$

$\lim_{n \to \infty} \frac{\varepsilon \ln(n)}{\varepsilon n^{\varepsilon}} = \lim_{n \to \infty} \frac{1}{\varepsilon} \frac{\ln(n^{\varepsilon})}{n^{\varepsilon}}$ (→ 0)

$= \frac{1}{\varepsilon} \cdot 0 = 0.$

Example: Give the limit (if it exists) of $\left\{\frac{\ln(n+1)}{n}\right\}_{n=1}^{\infty} \to 0$

it's "like" $\frac{\ln(n)}{n}$.

$$\frac{\ln(n+1)}{n} = \underbrace{\frac{n+1}{n}}_{\to 1} \cdot \underbrace{\frac{\ln(n+1)}{n+1}}_{\to 0} \to 0.$$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{n} = 0$$

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Example: Give the limit (if it exists) of $\left\{\frac{3^n}{4^n}\right\}_{n=1}^{\infty} \to 0$

Note: $\frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n \to 0$

$\left|\frac{3}{4}\right| < 1$

$$\lim_{n\to\infty} \frac{3^n}{4^n} = 0.$$

Example: Give the limit (if it exists) of $\left\{\frac{2^{-n}}{3^{-n}}\right\}_{n=1}^{\infty}$ Diverges

$$\frac{2^{-n}}{3^{-n}} = \left(\frac{2}{3}\right)^{-n} = \left(\frac{3}{2}\right)^n \to \infty$$

$\frac{3}{2} > 1$

$\lim_{n\to\infty} \frac{2^{-n}}{3^{-n}} = \infty \Rightarrow$ the limit DNE.

Example: Give the limit (if it exists) of $\left\{n^{\frac{1}{n+2}}\right\}_{n=1}^{\infty} \to 1$

$$n^{\left(\frac{1}{n+2}\right)} = e^{\ln\left(n^{\frac{1}{n+2}}\right)}$$

$$= e^{\frac{1}{n+2}\ln(n)} = e^{\overbrace{\frac{\ln(n)}{n+2}}^{\to 0}}$$

$$\to e^0 = 1.$$

Note: Be careful $\left(2^n\right)^{1/n} \not\to 1$

$\parallel$

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Example: Give the limit (if it exists) of $\left\{\left(1-\frac{1}{n}\right)^n\right\}_{n=1}^{\infty} \to \frac{1}{e}$

recall: $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

$\therefore \left(1-\frac{1}{n}\right)^n = \left(1+\frac{-1}{n}\right)^n \to e^{-1}$

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3. 0

Example: Give the limit (if it exists) of $\left\{\left(1+\frac{3}{n}\right)^n\right\}_{n=1}^{\infty} \to e^3$

State whether the sequence converges and, if it does, find the limit.

2^n. Diverges	$\frac{2}{n}. \to 0$	$\frac{(2n+1)^2}{(3n-1)^2}. = \left(\frac{2n+1}{3n-1}\right)^2 \to \frac{4}{9}$	$\ln\left(\frac{2n}{n+1}\right). \to \ln(2)$
$\frac{(-1)^n}{n}. \to 0$	$\sqrt{n}$. Diverges	$\frac{n^2}{\sqrt{2n^4+1}}. \to \frac{1}{\sqrt{2}}$	$\frac{n^4-1}{n^4+n-6}. \to 1$
$\frac{n-1}{n}. \to 1$	$\frac{n+(-1)^n}{n}. \to 1$	$\cos n\pi. = (-1)^n$ diverges	$\frac{n^5}{17n^4+12}.$ diverges
$\frac{n+1}{n^2}. \to 0$	$\sin\frac{\pi}{2n}. \to 0$	$e^{1/\sqrt{n}}. = e^{\frac{1}{\sqrt{n}}} \to 1$	$\sqrt{4-\frac{1}{n}}. \to 2$
$\frac{2^n}{4^n+1}. \to 0$	$\frac{n^2}{n+1}.$ diverges	$\ln n - \ln(n+1). \to 0$	$\frac{2^n-1}{2^n}. \to 1$
$(-1)^n\sqrt{n}.$ diverges	$\frac{4n}{\sqrt{n^2+1}}. \to 4$	$\frac{\sqrt{n+1}}{2\sqrt{n}}. \to \frac{1}{2}$	$\frac{1}{n}-\frac{1}{n+1}. \to 0$
$(-\frac{1}{2})^n. \to 0$	$\frac{4^n}{2^n+10^6}.$ diverges	$\left(1+\frac{1}{n}\right)^{2n} \to e^2$	$\left(1+\frac{1}{n}\right)^{n/2} \to e^{1/2}$
$\tan\frac{n\pi}{4n+1}. \to \tan(\pi/4) = 1$	$\frac{10^{10}\sqrt{n}}{n+1}. \to 0$	$\frac{2^n}{n^2}.$ diverges	$2\ln 3n - \ln(n^2+1). \to \ln(9)$